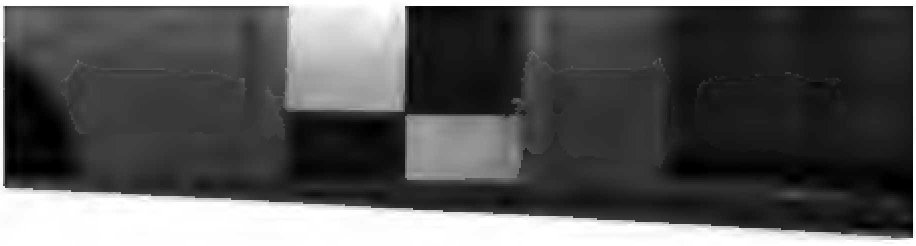
ROBINSON'S NEW HIGHER ARITHMETIC



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ROBINSON'S

NEW

HIGHER ARITHMETIC

FOR

HIGH SCHOOLS, ACADEMIES, AND
MERCANTILE COLLEGES

NEW YORK ·:· CINCINNATI ·:· CHICAGO
AMERICAN BOOK COMPANY

ROBINSON'S NEW ARITHMETICS

Graded to the wants of Primary, Intermediate, Grammar, Normal, and High Schools, Academies, and Colleges.

NEW PRIMARY ARITHMETIC.
NEW INTELLECTUAL ARITHMETIC.
NEW RUDIMENTS OF ARITHMETIC.
NEW PRACTICAL ARITHMETIC.
KEY TO NEW PRACTICAL ARITHMETIC.
NEW HIGHER ARITHMETIC.

ROBINSON'S ARITHMETICS have for many years had a wide use among the best American schools, and the new edition here announced preserves, in a new dress, with careful editing and revision, the leading features which have proved so satisfactory both to teacher and to pupil.

ROB.'S NEW HIGHER AR.

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PREFACE.

Robinson's Higher Arithmetic has proved its value and utility by a thirty-five years' test, and its plan and features are so well known that it seems unnecessary to call attention to them. The present edition has been thoroughly revised and reset. Nothing of value or merit in the old edition has been discarded, but it has been the aim of the revision to completely *modernize* the book and to make it as fully up to date as any of the most recent books published.

While almost every chapter has been entirely rewritten, and much new and valuable matter has been added, the general plan of the book has not been disturbed.

The work in its present form is intended to complete a well graded and progressive series of Arithmetics, and to furnish to *advanced students* a full and comprehensive text-book on the science of numbers — a work that shall embrace those subjects necessary to give the pupil a thoroughly practical and scientific arithmetical education.

As the book is addressed to advanced pupils, the author has throughout assumed that the student has an elementary knowledge of the fundamental principles of arithmetic. With this as a basis it has been possible to treat the subjects in a much more logical and scientific way than would be practicable in an elementary work.

The following characteristics of a first class text-book will be obvious to all who examine this work: the philosophical and scientific arrangement of the subjects; clear and concise definitions; full and rigid analyses; exact and comprehensive rules; brief and accurate methods of operation; the wide range of subjects and the large number and practical character of the examples — in a word, *scientific accuracy* combined with *practical utility*, throughout the work.

Much labor and attention have been devoted to obtaining correct and adequate information pertaining to mercantile and commercial transactions, and the Government Standard units of measures, weights, and money. The counting-room, the bank, the insurance and broker's office, the navy and ship-yard, the manufactory, the wharves, the custom-house, and the mint, have all been visited, and the most reliable statistics and the latest statutes have been consulted, for the purpose of securing *entire accuracy* in those parts of this work which relate to these subjects and departments.

By treating the decimal notation in the very beginning of the book as part of the general scheme of Arabic notation, it has been possible, at an early stage, to introduce simple practical examples which, in most books, do not find a place till near the end of the work. Thus, as applications of addition and subtraction will be found examples in bookkeeping; under multiplication and division examples in accounts and bills, surfaces and volumes, involution, as well as simple examples in interest, taxes, insurance, exchange, etc.

Special attention is called to the treatment of the Metric System; and to the chapter on Practical Measurements, in which many new subjects, such as temperature, specific gravity, etc., have been introduced.

The uniformity of plan in the book adds greatly to the ease with which the subjects are mastered. When once familiar with the arrangement the student knows immediately where to look for and

find any special point or detail in any chapter. First he finds a clear and logical development of the subject, then a model example with the solution, and lastly, the rule.

It is hoped that this book in its new guise will meet the demands of all who require a thorough, scientific training in higher arithmetic — such a training as will enable the student to solve, without difficulty, any example that may be given on an examination paper, as well as those that constantly present themselves for solution in the ordinary business of life.

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HIGHER ARITHMETIC.

GENERAL DEFINITIONS.

1. **Quantity** is any thing that can be increased, diminished, or measured; as distance, space, weight, motion, time.

2. A **Unit** is *one*, a single thing, or a definite quantity.

3. A **Number** is a unit, or a collection of units.

4. The **Unit of a Number** is *one* of the collection constituting the number. Thus, the unit of 34 is 1; of 34 days it is 1 day.

5. An **Abstract Number** is a number used without reference to any particular thing or quantity; as 3, 24, 756.

6. A **Concrete Number** is a number used with reference to some particular thing or quantity; as 3 hours, 24 cents, 756 miles.

7. **Unity** is the unit of an abstract number.

8. The **Denomination** is the name of the unit of a concrete number; as 1 *apple*, 1 *mile*.

9. A **Simple Number** is either an abstract number, or a concrete number of but one denomination; as 48, 52 pounds, 36 days.

10. A **Compound Number** is a concrete number expressed in two or more denominations; as 4 bushels 3 pecks; 8 rods 4 yards; 2 feet 3 inches.

11. An **Integral Number**, or an **Integer**, is a number which expresses whole things; as 5, 12 dollars, 17 men.

12. A **Fractional Number**, or a **Fraction**, is a number which expresses equal parts of a whole thing or quantity; as $\frac{1}{2}$, $\frac{3}{4}$ of a pound, $\frac{5}{12}$ of a bushel.

NOTE.—The *Denominator* of the fraction names the number of parts into which the unit is divided, and the *Numerator* shows how many of these parts are taken in the fraction. Thus, in $\frac{3}{4}$, 4 is the denominator and 3 the numerator.

13. **Decimals** are parts obtained by dividing an integer into tenths, hundredths, thousandths, etc.; as $\frac{3}{10}$ or .3, .06 of a mile.

14. **Like Numbers** have the same kind of unit, or express the same kind of quantity. Thus, 74 and 16, or 74 pounds and 16 pounds, are like numbers; also 4 weeks and 3 days, both being used to express units of time.

15. **Unlike Numbers** have different kinds of units, or are used to express different kinds of quantity. Thus, 36 miles and 15 days; 5 hours and 5 cents, are unlike numbers.

16. A **Scale** is a series of units, increasing or decreasing by fixed multipliers or divisors. Thus, 10, 100, 1000, 10000; 10000, 1000, 100, 10; 12, 144, 1728, are scales.

17. A **Uniform Scale** is one in which the multiplier or divisor is the same throughout the entire succession of units; as 10, 100, 1000, 10000; 5, 25, 125, 625.

18. A **Varying Scale** is one in which the multiplier or divisor is not the same throughout the entire succession of units; as the scale of linear measure, 1 in. 1 ft. 1 yd., etc., in which the multiplier changes, being 12 from the first unit to the second, 3 from the second to the third, etc.

19. A **Decimal Scale** is one in which the multiplier or divisor is uniformly ten; as 10, 100, 1000, 10000, etc.

20. **Mathematics** is the science of quantity.

21. The two fundamental branches of Mathematics are **Geometry** and **Arithmetic**. Geometry considers quantity with reference to position, form, and extension. Arithmetic considers quantity as an assemblage of definite *portions*, and treats *only of those conditions* and attributes which may be investigated *and expressed by* numbers.

22. **Arithmetic** is the *science* of numbers, and the *art* of computation.

NOTE. — When Arithmetic treats of operations on abstract numbers it is a science, and is called *Pure Arithmetic*. When it treats of operations on concrete numbers it is an art, and is called *Applied Arithmetic*. Pure and Applied Arithmetic are also called *Theoretical* and *Practical Arithmetic*.

23. An **Equation** is an expression of equality between two numbers or combinations of numbers.

24. A **Problem** is a question to be solved by computation.

25. A **Principle** is a general truth taken as a basis for a computation.

26. A **Rule** is a general direction for solving problems of a particular kind.

27. A **Demonstration** is a process of reasoning by which a truth or principle is established.

28. A **Solution** is a statement showing how a problem is solved.

29. **Analysis**, in Arithmetic, is the process of investigating principles, and solving problems, independently of set rules.

30. An **Operation** is a process in which figures are employed to make a computation, or to obtain some arithmetical result.

31. The **Five Fundamental Operations** of Arithmetic are, Notation and Numeration, Addition, Subtraction, Multiplication, and Division.

SIGNS.

32. The following are the signs used in Arithmetic:

Addition	$+$	Involution		3^3
Subtraction	$-$	Evolution	Square root	$\sqrt{}$
Multiplication	$\times$		Cube root	$\sqrt[3]{}$
Division	$\div$ $\overline{)}$	Parentheses		( )
Equality	$=$	Vinculum		——
Ratio	:	Brackets		[ ]
Proportion	: :	Braces		{ }

NOTE. — Numbers included in parentheses, brackets, or under the vinculum are to be considered together and subjected to the same operation. Thus, $(8+4)\times5$, $\overline{8+4}\times5$, $[8+4]\times5$ and $\{8+4\}\times5$ indicate that the sum of 8 and 4, 12, is to be multiplied by 5, while $8+4\times5$ means that 8 is to be added to 4×5, or 20.

AXIOMS.

33. An **Axiom** is a self-evident truth. The principal axioms required in arithmetical investigations are the following:

1. *If the same quantity or equal quantities are added to equal quantities, the sums will be equal.*

Thus, $3 + 1 = 4$ and $3 + 1 + 2 = 4 + 2$.

2. *If the same quantity or equal quantities are subtracted from equal quantities, the remainders will be equal.*

Thus, $3 + 1 = 4$ and $3 + 1 - 2 = 4 - 2$.

3. *If equal quantities are multiplied by the same number, the products will be equal.*

Thus, $3 + 1 = 4$ and $(3 + 1) \times 2 = 4 \times 2$.

4. *If equal quantities are divided by the same number, the quotients will be equal.*

Thus, $3 + 1 = 4$ and $(3 + 1) \div 2 = 4 \div 2$.

5. *If the same number is added to a quantity and subtracted from the sum, the remainder will be that quantity.*

Thus, $3 + 1 = 4$ and $4 - 1 = 3$; or $3 + 1 - 1 = 3$.

6. *If a quantity is multiplied by a number and the product divided by the same number, the quotient will be that quantity.*

Thus, $3 \times 4 = 12$ and $12 \div 4 = 3$; or $3 \times 4 \div 4 = 3$.

7. *Quantities which are respectively equal to any other quantity are equal to each other.*

Thus, $8 \times 2 = 16$ and $4 \times 4 = 16$; hence $8 \times 2 = 4 \times 4$.

8. *Like powers or like roots of equal quantities are equal.*

Thus, $3 + 1 = 4$ and $(3 + 1)^2 = 4^2$; $\sqrt{3 + 1} = \sqrt{4}$.

9. *The whole of any quantity is greater than any of its parts.*

Thus, $2 + 3 + 1 + 4 = 10$ and 10 is greater than 2, 3, 1, or 4.

10. *The whole of any quantity is equal to the sum of all its parts.*

Thus, $2 + 3 + 5 + 6 = 16$ and 16 is equal to the sum of all the parts.

NOTATION AND NUMERATION.

34. Notation is a method of *writing* or expressing numbers by characters.

35. Numeration is a method of *reading* numbers expressed by characters.

36. Two systems of notation are in general use—the **Roman** and the **Arabic.**

NOTE.—The Roman notation is supposed to have been first used by the Romans, hence its name. The Arabic notation was first introduced into Europe by the Moors or Arabs, who conquered and held possession of Spain during the 11th century. It received the attention of scientific men in Italy at the beginning of the 18th century, and was soon afterward adopted in most European countries. Formerly it was supposed to be an invention of the Arabs; but investigations have shown that the Arabs adopted it from the Hindoos, among whom it has been in use more than 2000 years. From this undoubted origin it is sometimes called the *Indian notation*.

THE ROMAN NOTATION.

37. This system employs seven capital letters to express numbers. Thus,

LETTERS	I	V	X	L	C	D	M
VALUES	One	Five	Ten	Fifty	One Hundred	Five Hundred	One Thousand

38. The Roman notation is founded upon five principles, as follows:

PRINCIPLES.—I. *Repeating a letter repeats its value.*

Thus, II represents two, XX twenty, CCC three hundred.

II. *If a letter of any value is placed after one of greater value, its value is to be united to that of the greater.*

Thus, XI represents eleven, LX sixty, DC six hundred.

III. *If a letter of any value is placed before one of greater value, its value is to be taken from that of the greater.*

Thus, IX represents nine, XL forty, CD four hundred.

IV. *If a letter of any value is placed between two letters, each of greater value, its value is to be taken from the united value of the other two.*

Thus, XIV represents fourteen, XXIX twenty-nine, XCIV ninety-four.

V. *A bar or dash placed over a letter increases its value one thousand fold.*

Thus, V signifies five, and $\bar{V}$ five thousand; L fifty, and $\bar{L}$ fifty thousand.

39. The following table shows the method of expressing numbers by Roman notation:

TABLE OF ROMAN NOTATION.

I = One.	XX = Twenty.
II = Two.	XXI = Twenty-one.
III = Three.	XXX = Thirty.
IV = Four.	XL = Forty.
V = Five.	L = Fifty.
VI = Six.	LX = Sixty.
VII = Seven.	LXX = Seventy.
VIII = Eight.	LXXX = Eighty.
IX = Nine.	XC = Ninety.
X = Ten.	C = One hundred.
XI = Eleven.	CC = Two hundred.
XII = Twelve.	D = Five hundred.
XIII = Thirteen.	DC = Six hundred.
XIV = Fourteen.	M = One thousand.
XV = Fifteen.	MC = One thousand one hundred.
XVI = Sixteen.	MM = Two thousand.
XVII = Seventeen.	$\bar{X}$ = Ten thousand.
XVIII = Eighteen.	$\bar{C}$ = One hundred thousand.
XIX = Nineteen.	$\bar{M}$ = One million.

NOTES.—1 Though the letters used in the above table have been employed as the Roman numerals for many centuries, the marks or characters used originally in this notation were as follows:

Primitive characters,	I	V	X	L	C	D	M
Modern numerals,	I	V	X	L	C	D	M

2. The system of Roman notation is not well adapted to the purposes of numerical calculation; it is principally confined to the numbering of chapters and sections of books, *public documents*, prescriptions for medicines, the numbers on clock dials, etc.

Examples.

40. Express the following numbers by the Roman notation:

1. Fourteen.
2. Nineteen.
3. Twenty-four.
4. Thirty-nine.
5. Forty-six.
6. Fifty-one.
7. Eighty-eight.
8. Seventy-three.
9. Ninety-five.
10. One hundred one.
11. Five hundred fifty-five.
12. One thousand three.
13. Twenty thousand eight hundred forty-five.
14. One hundred thousand seventy-seven.

THE ARABIC NOTATION.

41. This system employs ten characters or figures to express numbers.

FIGURES	0	1	2	3	4	5	6	7	8	9
NAMES AND VALUES	Naught or Cipher	One	Two	Three	Four	Five	Six	Seven	Eight	Nine

42. These figures and the numbers they express are called the ten **Digits.**

NOTE. — Some authorities do not include the 0 with the digits; they then have only nine.

43. The cipher, or first character, is called *naught,* because it has no value of its own. It is otherwise termed *nothing,* and *zero.* The other nine characters are called *significant figures,* because each has a value of its own.

44. The ten Arabic characters are the **Alphabet** of Arithmetic. Used independently, they can express only the nine numbers that correspond to the names of the figures 1 to 9. But when combined according to certain principles, they serve to express all numbers.

45. The notation of all numbers by the ten figures is accomplished by the formation of a series of units of different values, to which the digits may be successively applied. First, ten

simple units are considered together, and treated as a single superior unit; then, a collection of ten of these new units is taken as a still higher unit; and so on, indefinitely. A regular series of units, in ascending orders, is thus formed:

TABLE OF UNITS.

Primary units are called	units of the first order.
Ten units of the first order make	1 unit of the second order.
Ten units of the second order make	1 unit of the third order.
Ten units of the third order make	1 unit of the fourth order.

46. The various orders of units, when expressed by figures, are distinguished from each other by their *location*, or the place they occupy in a horizontal row of figures. Units of the first order are written at the right hand; units of the second order occupy the second place; units of the third order, the third place; and so on, counting from right to left.

ORDERS.	NAMES.	ORDERS.	NAMES.
1st	Ones or Units.	8th	Ten Millions.
2d	Tens.	9th	Hundred Millions.
3d	Hundreds.	10th	Billions.
4th	Thousands.	11th	Ten Billions.
5th	Ten Thousands.	12th	Hundred Billions.
6th	Hundred Thousands.	13th	Trillions.
7th	Millions.		

13th	12th	11th	10th	9th	8th	7th	6th	5th	4th	3d	2d	1st
3	4	5	8	9	0	6	0	2	1	4	3	6

In this notation we observe that—

1. A figure written in the place of any order expresses as many units of that order as are denoted by the name of the figure used. Thus, 436 expresses 4 units of the 3d order, 3 units of the 2d order, and 6 units of the 1st order.

2. The cipher, having no value of its own, is used to fill the places of vacant orders, and thus preserve the relative positions of the significant figures. Thus, in 50, the cipher shows the absence of simple units, and at the same time gives to the figure 5 the local value of the second order of units.

47. Orders to the right of units represent divisions of a unit into *tenths, hundredths, thousandths,* etc., and are called **Decimals.**

The first order to the left of units, as we have seen, is *tens;* the first order to the right of units is *tenths;* thus, 1.5 is one and five *tenths.* The second order to the right of units is *hundredths;* the third order *thousandths,* and so on. All orders at equal distances to the right and left of *units* have corresponding names.

DECIMAL ORDERS.	NAMES.	DECIMAL ORDERS.	NAMES.
1st . . .	Tenths.	7th . . .	Ten-millionths.
2d . . .	Hundredths.	8th . . .	Hundred-millionths.
3d . . .	Thousandths.	9th . . .	Billionths.
4th . . .	Ten-thousandths.	10th . . .	Ten-billionths.
5th . . .	Hundred-thousandths.	11th . . .	Hundred-billionths.
6th . . .	Millionths.	12th . . .	Trillionths.

48. Since the number expressed by any figure depends upon the place it occupies, figures have two values, simple and local.

49. The **Simple Value** of a figure is its value when taken alone; thus, 4, 7, 2.

50. The **Local Value** of a figure is its value when used with another figure or figures in the same number. Thus, in 325.4 the local value of the 3 is 300, of the 2 is 20, of the 5 is 5 units, and of the 4 is 4 tenths.

NOTE.—When a figure occupies units' place, its simple and local values are the same.

51. The units of the various orders of a number are sometimes called the **Terms** of a number. Thus the number 325 has three terms; 5 is the units' term, 2 the tens' term, and 3 the hundreds' term.

NOTE.—The word *term* is applied to the *figures* as well as to the *numbers* they express.

52. The leading principles upon which the Arabic notation is founded are embraced in the following laws:

GENERAL LAWS.—I. *All numbers are expressed by applying the ten figures to different orders of units.*

II. *The different orders of units increase from right to left, and decrease from left to right, in a tenfold ratio.*

III. ***Every removal of a figure one place to the left, increases its local value tenfold; and every removal of a figure one place to the right, diminishes its local value tenfold.***

IV. ***Numbers to the right of the units' place increase and decrease precisely like those to the left, and are separated from the units by a decimal point.***

53. Our method of numerating, or naming, groups the successive orders into *periods* of three figures each, there being three orders of thousands, three orders of millions, and so on in all higher orders. These periods are sometimes separated by commas, as in the following table:

NUMERATION TABLE.

Place	Figure	Order	Period
Hundreds of Trillions.	1	15th order,	5th period.
Tens of Trillions.	6	14th order,	
Trillions,	5,	13th order,	
Hundreds of billions.	9	12th order,	4th period.
Tens of billions.	8	11th order,	
Billions	3,	10th order,	
Hundreds of millions	2	9th order,	3d period.
Tens of millions.	1	8th order,	
Millions.	5,	7th order,	
Hundreds of thousands	7	6th order,	2d period.
Tens of thousands.	3	5th order,	
Thousands.	2,	4th order,	
Hundreds.	7	3d order,	1st period.
Tens	5	2d order,	
Ones or Units.	4	1st order,	
Decimal sign.	.		
Tenths	5	1st order,	Decimals.
Hundredths.	7	2d order,	
Thousandths.	3,	3d order,	
Ten-thousandths.	2	4th order,	
Hundred-thousandths	5	5th order,	
Millionths.	6,	6th order,	
Ten-millionths.	3	7th order,	
Hundred-millionths.	7	8th order,	
Billionths.	8	9th order,	

NOTE. — The above is the French method of numerating, and is the one in general use in this country. The English numerate by periods of six figures each. See § 60.

54. The denominator of a decimal fraction, when expressed, is necessarily 10, 100, 1000, or some power of 10. By examining the table it will be seen that the number of places in a decimal is equal to the number of ciphers required to express its denominator. Thus, tenths occupy the first place at the right of units, and the denominator of $\frac{1}{10}$ has one cipher; hundredths in the table extend two places from units, and the denominator of $\frac{1}{100}$ has two ciphers; and so on.

55. A decimal is usually read as expressing a certain number of decimal units of the *lowest order* contained in the decimal. Thus, 5 tenths and 4 hundredths, or .54, is read, fifty-four hundredths.

The currency of the United States is decimal, since 10 mills make 1 cent, 10 cents 1 dime, and 10 dimes 1 dollar. Dollars are written to the left of the decimal point. The first two places at the right of the decimal point express dimes and cents, or tens and units of cents, and the third place, mills. Thus: Twenty-five dollars, sixty-three cents, four mills, is written $25.634; One dollar and five cents is written $1 05. A mill is simply an expression for a tenth of a cent, for there is no such coin as the mill in use.

56. The names of the *periods* are derived from the Latin numerals. The twenty-two given below extend the numeration table to the sixty-sixth place or order, inclusive.

PERIODS.	NAMES.	PERIODS.	NAMES.
1st . . .	Units.	12th . . .	Decillions.
2d . . .	Thousands.	13th . . .	Undecillions.
3d . . .	Millions.	14th . . .	Duodecillions.
4th . . .	Billions.	15th . . .	Tredecillions.
5th . . .	Trillions.	16th . . .	Quatuordecillions.
6th . . .	Quadrillions.	17th . . .	Quindecillions.
7th . . .	Quintillions.	18th . . .	Sexdecillions.
8th . . .	Sextillions.	19th . . .	Septendecillions.
9th . . .	Septillions.	20th . . .	Octodecillions.
10th . . .	Octillions.	21st . . .	Novendecillions.
11th . . .	Nonillions.	22d . . .	Vigintillions.

57. RULE FOR NOTATION.—I. *Beginning at the left hand, write the figures that express the hundreds, tens, and units of each successive period in their order, placing a cipher wherever an order of units is omitted.*

II. *Write the decimal point at the right of the units and after it the figures that express the units of the successive decimal orders.*

58. RULE FOR NUMERATION.—I. *Separate the whole number into periods of three figures each, commencing at units.*

II. *Beginning at the left hand, read each period separately, and give the name to each period, except the last, or period of units.*

III. *Read the decimal as an integer, and give to it the name of its lowest or right-hand order.*

NOTE. — In a mixed number composed of an integer and a decimal, read the word *and* between the integer and the decimal.

Examples.

59. Write and read the following numbers:

1. One unit of the third order, two of the second, five of the first.

2. Two units of the fifth order, four of the fourth, five of the second, six of the first.

3. Seven units of the fourth order, five of the third, three of the second, eight of the first.

4. Nine units of the fourth order, two of the third, four of the first.

5. Five units of the fourth order, eight of the second.

6. Five units of the fifth order, one of the third, eight of the first.

7. Three units of the fifth order, six of the fourth, four of the third, seven of the first.

8. Eight units of the first decimal order, three of the second.

9. Two units of the sixth order, four of the fifth, nine of the fourth, three of the third, five of the first.

10. Three units of the eighth order, five of the seventh, four of the sixth, three of the fifth, eight of the fourth, five of the third, eight of the second, seven of the first.

11. Three units of the ninth order, eight of the seventh, four of the sixth, six of the fifth, nine of the first.

12. Five units of the twelfth order, three of the eleventh, six of the tenth.

13. Four units of the twelfth order, five of the tenth, eight of the fifth, nine of the fourth, four of the third, and nine of the first decimal order, four of the second, six of the third, eight of the fourth.

14. Three units of the second decimal order, five of the third, eight of the fifth.

15. Seven units of the first decimal order, eight of the third, nine of the fifth, seven of the sixth.

16. Eight units of the sixth order, three of the fifth, two of the fourth, three of the third, eight of the first; and nine of the second decimal order, two of the third, eight of the fifth, three of the sixth.

Write the following numbers in figures:

17. Forty-eight.

18. One hundred sixty-four.

19. Forty-eight thousand seven hundred eighty-nine.

20. Five hundred thirty-six million three hundred forty-seven thousand nine hundred seventy-two.

21. Ninety-nine billion thirty-seven thousand four.

22. Eight hundred sixty-four billion five hundred thirty-eight million two hundred seventeen thousand nine hundred fifty-three.

23. Five tenths.

24. Thirty-six hundredths.

25. Seventy-five ten-thousandths.

26. 6 hundred-millionths; 600 millionths.

27. Three hundred and fourteen thousand eight hundred and seventy-seven millionths.

28. Two dollars and fifty-three cents.

29. Five thousand twenty dollars and five cents.

30. Sixty-two thousand four hundred fifteen dollars 25 cents 5 mills.

31. Two hundred thirty-five trillion one hundred four billion seven hundred fifty million sixty-six thousand ten and twenty-five hundredths.

32. Seven hundred forty-one trillion fifty-four billion one hundred eleven million one hundred one and four hundred ninety-six thousandths.

33. Twelve trillion fourteen billion three hundred sixty million and five thousand twenty-nine hundred-thousandths.

Write the following numbers in figures, and read them:

34. Twenty-five units in the second period, four hundred ninety-six in the first.

35. Three hundred sixty-four units in the third period, seven hundred fifteen in the second, eight hundred thirty-two in the first.

36. Eighty-one units in the fifth period, two hundred nineteen in the fourth, fifty-six in the third.

37. Nine hundred forty-five units in the seventh period, eighteen in the fifth, one hundred three in the third, sixteen in the second, eight in the first.

38. One unit in the tenth period, five hundred thirty-six in the ninth, two hundred forty-seven in the eighth, nine hundred twenty-four in the seventh, three hundred twelve in the sixth, fourteen in the fourth, two in the first.

Read the following numbers:

39. 24.5.	**44.** 54321.	**49.** 247843112.
40. 564.2.	**45.** 2005.5.	**50.** 2367854.278.
41. .075.	**46.** $ 2483.50.	**51.** $ 5300008.063.
42. 2.503.	**47.** .001007.	**52.** 10100500.309.
43. .0725.	**48.** $ 1003.34.	**53.** 75000040037.

60. The English method of notation as used in Great Britain, divides numbers by periods of *six* figures each as shown below: —

Hundred-thousands.	Ten-thousands.	Thousands.	Hundreds.	Tens.	Ones or Units.	Hundred-thousands.	Ten-thousands.	Thousands.	Hundreds.	Tens.	Ones or Units.	Hundred-thousands.	Ten-thousands.	Thousands.	Hundreds.	Tens	Ones or Units.	Hundred-thousands.	Ten-thousands.	Thousands.	Hundreds.	Tens.	Ones or Units.
1	8	7	3	4	5,	9	3	2	4	1	0,	3	1	8	0	1	2,	5	0	1	0	9	2
4th Period Trillions						3d Period Billions						2d Period Millions						1st Period Ones or Units					

SCALES OF NOTATION.

61. In any uniform scale (§ 17) the number of units of one order required to make one of the next higher order is called the **Radix** of the scale.

62. In the foregoing decimal system of notation the radix is 10. Although the decimal system of notation has been found very convenient, numbers can be expressed in a number of other uniform scales. The following table gives the names and radixes of several of these.

NAME.	RADIX.	NAME.	RADIX.
Binary	2	Septenary	7
Ternary	3	Octary	8
Quaternary	4	Nonary	9
Quinary	5	Undenary	11
Senary	6	Duodecimal	12

NOTE. — In like manner any number, as 25 or 97, might be taken as the basis of a scale.

63. A scale contains as many digits as there are units in its radix; and every scale must have the digit, 0.

NOTE. — The decimal scale has ten digits, 1, 2, 3, 4, 5, 6, 7, 8, 9, 0; the nonary has nine, 1, 2, 3, 4, 5, 6, 7, 8, 9; the octary eight, 1 to 7 and 0; the septenary seven, 1 to 6 and 0; the senary six, 1 to 5 and 0; the quinary five, 1 to 4 and 0; the quaternary four, 1 to 3 and 0; the ternary three, 1, 2, 0; the binary two, 1, 0; the undenary eleven, 1 to 9, and one more (which may be expressed by any character, as *a*), and 0; and the duodecimal twelve, 1 to 9, *a*, *b*, 0.

By combining the digits in any scale, we obtain the notation of that scale.

In the decimal scale when we pass 9, we must use two digits for the next number, which is 10. In like manner in the quaternary scale, when we pass 3, we must use two digits; therefore, 4 in the decimal scale is written 10 in the quaternary scale. In the quinary scale, when we pass 4, we must use two digits; therefore, 5 in the decimal scale is written 10 in the quinary. In the decimal scale, when we pass 99, we have to use three digits for the next number, which is 100; but the highest number we can write in the quinary scale in two figures is 44, which corresponds to 24 in the decimal scale. Therefore, 25 in the decimal scale would be written 100 in the quinary; and for the same reason 125 would be written 1000; 625, 10000, etc.

In the ternary scale, 3 would be written 10, 9 would be written 100, 27 would be written 1000; 81, 10000, etc. In the senary scale, 6 would be written 10, 36 would be 100, 216 would be 1000, 1296 would be 10000, etc.

64. The following table shows the notation in the various scales with the corresponding value of the numbers, in the decimal scale:

DECIMAL.	1	2	3	4	5	6	[illegible]	8	9	10	11	12	13	14
Binary	1	10	11	100	101	110	111	1000	1001	1010	1011	1100	1101	1110
Ternary . . .	1	2	10	11	12	20	21	22	100	101	102	110	111	[illegible]
Quaternary .	1	2	3	10	11	12	13	20	21	22	[illegible]	30	31	[illegible]
Quinary . . .	1	2	[illegible]	4	10	11	12	13	14	20	21	22	23	24
Senary	1	2	3	4	5	10	11	12	13	[illegible]	15	[illegible]	21	22
Septenary . .	1	2	3	4	5	6	10	11	12	13	14	15	16	20
Octary	1	2	[illegible]	4	5	6	7	10	11	12	13	[illegible]	15	16
Nonary	1	2	3	4	[illegible]	6	7	8	10	11	12	13	14	15
Undenary . .	1	2	3	4	5	6	7	[illegible]	9	a	10	11	12	[illegible]
Duodecimal .	1	2	3	4	5	6	7	8	9	a	b	10	11	[illegible]

DECIMAL.	15	16	[illegible]	18	19	20	21	22	23
Binary	1111	10000	10001	10010	10011	10100	10101	10110	10111
Ternary . . .	120	121	122	200	201	[illegible]	[illegible]	211	212
Quaternary .	33	100	101	102	103	110	111	112	113
Quinary . . .	30	31	32	33	34	[illegible]	41	[illegible]	43
Senary	23	24	25	30	31	32	33	34	[illegible]
Septenary . .	21	22	23	24	25	[illegible]	30	31	32
Octary	17	20	21	22	23	[illegible]	25	26	27
Nonary . . .	16	17	[illegible]	20	21	22	23	24	25
Undenary . .	14	15	16	17	18	19	1a	20	21
Duodecimal .	13	14	15	16	17	18	19	1a	[illegible]

DECIMAL.	24	25	26	[illegible]	28	29	30	100
Binary	11000	11001	11010	11011	11100	11101	11110	1100100
Ternary . . .	220	221	222	1000	1001	1002	1010	10201
Quaternary .	120	121	122	123	130	131	132	1210
Quinary . . .	44	100	101	102	103	104	110	400
Senary	40	41	[illegible]	43	44	45	[illegible]	244
Septenary . .	33	34	35	36	40	41	42	202
Octary	30	31	32	33	34	35	36	144
Nonary	26	27	28	30	31	32	33	121
Undenary . .	22	23	24	25	26	27	28	91
Duodecimal .	20	21	[illegible]	23	24	25	26	84

NOTE.—For convenience the radix of a scale is indicated by a small figure placed below the right-hand order. Thus, 244 in the senary scale is indicated 244_6; 84 in the duodecimal 84_{12}.

Examples.

65. 1. Write in the duodecimal scale the numbers corresponding to those from 35 to 99 in the decimal scale.

2. Write in the undenary scale, the numbers from 31 to 75 in the decimal scale.

Write the numbers from 1 to 100 in the following scales:

3. Ternary.	5. Senary.	7. Binary.	9. Septenary.
4. Quaternary.	6. Quinary.	8. Octary.	10. Nonary.

66. To change from the decimal scale to another scale.

1. Change 6239 from the decimal to the quinary scale.

OPERATION.

5)6239
5)1247 + 4 of 1st.
5)249 + 2 of 2d.
5)49 + 4 of 3d.
5)9 + 4 of 4th.
1 + 4 of 5th.
of 6th.

SOLUTION.—Since in the quinary scale 5 units of one order make 1 of the next higher, dividing 6239 by 5 we have 1247 units of the second order and four remaining in the first; dividing 1247 by 5 we have 249 units of the third order, and 2 remaining in the second. Continuing the division until no number equal to or greater than 5 remains in the quotient, we have one unit of the sixth order, 4 of the fifth, 4 of the fourth, 4 of the third, 2 of the second, and 4 of the first, or 144424.

RULE.—*Divide the number in the decimal scale successively by the radix of the scale required, preserving each remainder until the quotient is less than the radix. The first remainder will represent units, the second tens, etc., and the last quotient will represent the highest order of the number in the required scale.*

NOTE.—If after any division there is no remainder, a cipher must be placed in the corresponding order.

2. Express in the binary scale: 236, 765, 1290.

3. Express in the quaternary scale: 495, 3256, 7598, 49200.

4. Express in the ternary, senary, and quinary scales: 6342, 7187, 8941.

5. Express in the septenary and octary scales: 41, 357, 2325.

6. Express in the nonary, undenary, and duodecimal scales: 29, 100, 229, 3568.

67. To change from any scale to the decimal scale.

1. Change 543_6 to the decimal scale.

OPERATION.

$$\begin{array}{r} 543 \\ \underline{6} \\ 34 \\ \underline{6} \\ 207 \; \textit{Ans.} \end{array}$$

SOLUTION.—Since a higher unit in the senary scale equals 6 of the next lower, 5 units of the third order equal 30 of the second, and 4 added = 34. In like manner 34 units of the second order = 204 of the first, + 3 = 207. Hence the number in the decimal scale is 207.

RULE.—*Beginning with the highest order, multiply each successive order by the radix of the scale, and add to the result the figure in the next lower order, until the lowest order is reached. The last result will be the number as expressed in the decimal scale.*

Change the following numbers to the decimal scale:

2. 11111_2.	6. 13041_6.	10. 31319_{11}.	14. 333333_4.
3. 10202_3.	7. 64350_7.	11. 47698_{12}.	15. 444444_5.
4. 12320_4.	8. 45761_8.	12. 100000_2.	16. 555555_6.
5. 32430_5.	9. 57684_9.	13. 222222_3.	17. 666666_7.

NOTE.—All operations can be performed in any scale on the same principles as in the decimal. It is only necessary to bear in mind each time, how many units of one order make one of the next higher.

68. Since varying scales do not increase by one constant multiplier, in working examples in such scales, we must bear in mind how many units of each order make one of the next higher.

Thus, in the scale, 1 pound, 1 ounce, 1 pennyweight, 1 grain, since 24 grains = 1 pennyweight, 20 pennyweights = 1 ounce, 12 ounces = 1 pound, it follows that 24 units of the first order make 1 of the second; 20 of the second, 1 of the third; and 12 of the third, 1 of the fourth. Hence, the multipliers are successively 24, 20, 12. In like manner in the scale of linear measure, since 12 inches = 1 foot, 3 feet = 1 yard, $5\frac{1}{2}$ yards = 1 rod, 40 rods = 1 furlong, and 8 furlongs = 1 mile, it follows that 12 units of the first order make 1 of the second; 3 of the second, 1 of the third; $5\frac{1}{2}$ of the third, 1 of the fourth; 40 of the fourth, 1 of the fifth; and 8 of the fifth, 1 of the sixth.

NOTE.—Varying scales are used chiefly in computations with compound denominate numbers.

ADDITION.

69. **Addition** is the process of uniting *several* numbers of the same kind into *one* equivalent number.

70. The **Addends** are the numbers to be added.

71. The **Sum** or **Amount** is the result obtained by the addition.

72. When the given numbers contain several orders of units, the method of addition is based upon the following principles:

PRINCIPLES. — I. *If the like orders of units are added separately, the sum of all the results must be equal to the entire sum of the given numbers.* (Ax. 10.)

II. *If the sum of the units of any order contains units of a higher order, these higher units must be combined with units of like order.*

III. *The work must commence with the lowest unit, in order to combine the partial sums in a single expression, at one operation.*

Examples.

73. 1. Find the sum of 397, 476, and 873.

OPERATION.

$$\begin{array}{r} 397 \\ 476 \\ 873 \\ \hline 1746 \end{array}\ \textit{Ans.}$$

SOLUTION. — We arrange the numbers so that units of like order stand in the same column. We then add the first, or right hand column, and find the sum to be 16 units, or 1 ten and 6 units; writing the 6 units under the column of units, we add the 1 ten to the column of tens, and find the sum to be 24 tens, or 2 hundreds and 4 tens; writing the 4 tens under the column of tens, we add the 2 hundreds to the column of hundreds, and find the sum to be 17 hundreds, or 1 thousand and 7 hundreds; writing the 7 hundreds under the column of hundreds, and the 1 in thousands' place, we have the entire sum, 1746.

2. Find the sum of 3.975, 476.5321 and 32.2765.

OPERATION.

3.975
476.5321
32.2765
512.7836 *Ans.*

SOLUTION. — We arrange the numbers so that units of like order stand in the same column. This brings the decimal points directly under one another. We add as before and place the decimal point in the result between tenths and units or directly under the decimal points in the numbers added.

RULE. — I. *Write the numbers to be added so that all the units of the same order stand in the same column; that is, units under units, tens under tens, tenths under tenths, hundredths under hundredths, etc.*

II. *Commencing at the right hand, add each column separately, and write the sum underneath, if it is less than ten.*

III. *If the sum of any column is ten or more than ten, write the unit figure only, and add the ten or tens to the next column.*

IV. *Write the entire sum of the last column.*

V. *If there are decimal places in the addends, place the decimal point in the result directly under those in the number added.*

NOTES. — 1. In adding, learn to pronounce the partial results without naming the *figures*, separately. Thus, in the first example, say 8, 9, 16; 8, 15, 24; 10, 14, 17.

2. When the sum of any column is greater than 9, the process of adding the tens to the next column is called *carrying*.

Add:

3.	4.	5.	6.
8635	1234567	67.05	24603.2341
2194	723456	123.653	298765.0009
7421	34565	4567.101	47321.8762
5063	45666	89093.9	58653.31
2196	333	654321.08	5376.421
1245	53090	1234567.756	340.008

7. $123 + 456 + 785 + 12 + 345 + 901 + 567 = ?$

8. $12345 + 67890 + 8763 + 347 + 1037 + 198760 = ?$

9. $172 + 4005 + 3761 + 20472 + 367012 + 19762 = ?$

10. Add .375, .24, .536, .78567, .4637, and .57439.

11. Add 5.3756, 85.473, 9.2, 46.37859, and 45.248377.

12. Add .5, .37, .489, .6372, .47856, and .02524.

13. What is the sum of thirty-seven thousand six, four hundred twenty-nine thousand nine, and two million thirty-six?

14. Add eight hundred fifty-six thousand nine hundred thirty-three, one million nine hundred seventy-six thousand eight hundred fifty-nine, two hundred three million eight and one hundred ninety-five thousand seven hundred fifty-two.

15. What is the sum of one hundred sixty-seven thousand, three hundred sixty-seven thousand, nine hundred six thousand, two hundred forty-seven thousand, seventeen thousand, one hundred six thousand three hundred, forty thousand forty-nine, and ten thousand four hundred one?

16. What is the sum of 137 thousandths, 435 thousandths, 836 thousandths, 937 thousandths, and 496 thousandths?

17. What is the sum of one hundred two ten-thousandths, thirteen thousand four hundred twenty-six hundred-thousandths, five hundred sixty-seven millionths, three millionths, and twenty-four thousand seven hundred-thousandths?

74. To add two or more columns at once.

1. What is the sum of 4632, 2553, 4735, and 2863?

OPERATION.

$$\begin{array}{r} 4632 \\ 2553 \\ 4735 \\ 2863 \\ \hline 14783 \end{array}\ \textit{Ans.}$$

SOLUTION. — Beginning with the units and tens of the number last written, we add first the tens above, then the units, thus: 63 and 30 are 93, and 5 are 98, and 50 are 148, and 3 are 151, and 30 are 181, and 2 are 183. Of this sum, we write the 83 under the columns added, and carry the 1 to the next columns, thus: 28 and 1 are 29, and 40 are 69, and 7 are 76, and 20 are 96, and 5 are 101, and 40 are 141, and 6 are 147, which we write in its place, and we have the whole amount, 14783.

RULE. — *Taking the tens and units of the last addend together add to it first the tens and then the units of the next number; to this result add first the tens and then the units of the next number, and so on. Write the tens and units of the result in their proper places, and carry the higher orders to the next column. Proceed in the same way with the next two columns, and so on until all the columns are added.*

Find the sum, adding two columns at a time:

2.	3.	4.	5.
8450	75634	1234.56	$ 73490.42
5425	86213	470.21	28219.86
8595	92045	821.76	16218.73
6731	73461	5709.14	2367.19
7963	34719	3796.23	4019.63
5143	26054	75.42	672.54
4561	19732	253.20	450.67
6783	84160	576.44	9107.32
4746	97013	9081.76	63284.19
2373	34567	734.09	14376.51
3021	43651	31.47	97164.20
7273	52170	670.39	31912.32

75. Bookkeeping is the art of recording business transactions in a systematic manner.

Business records or accounts are kept in account books, and the chief of these are a daybook, a journal, and a ledger. Each account is recorded by item at the time it is made in the **Daybook**. A condensed statement of these accounts is recorded at the end of each day in the **Journal**. The **Ledger** is the principal book into which are brought into summary form the accounts from the journal or daybook. The items often form long columns, and accountants in adding usually add more than one column at a single operation.

Add the following ledger accounts:

1.	2.	3.	4.
$ 13.05	$ 300.25	$ 72.51	$ 1865.00
69.52	165.18	208.09	21009.00
321.80	2.01	3106.24	3450.00
652.09	31.15	2398.00	.56
15.38	208.99	100.09	2.25
5.25	.50	.75	300.00
16.00	.18	80.09	1800.00
105.00	18.56	275.00	21632.50
162.08	201.29	3500.00	42561.00
13.51	3.82	6000.00	3115.67

5.	6.	7.	8.
$42.17	$506.76	$2371.67	$14763.84
36.24	194.32	4571.84	33276.90
18.42	427.90	1690.50	47061.39
10.71	173.26	2037.69	18242.76
194.30	71.32	5094.46	37364.96
347.16	39.46	876.54	8410.31
40.00	152.60	679.81	5724.27
12.94	271.78	155.48	56317.66
86.73	320.00	4930.71	81742.72
271.19	709.08	3104.13	22431.27
103.07	48.50	1987.67	40163.55
500.50	63.41	5142.84	32189.60
7.59	56.00	276.30	7063.21
11.44	410.10	522.71	3451.09
81.92	372.22	3114.60	9200.00
110.10	137.89	1776.82	1807.36
107.09	276.44	7152.91	56768.72
207.16	18.19	9328.42	63024.27
97.20	27.96	472.19	36180.45
21.77	157.16	321.42	90807.08
150.15	94.57	2423.79	28763.81
427.26	177.66	1600.81	37196.75
316.42	327.40	5976.27	4230.61
114.64	1132.16	4318.19	3719.84
81.13	876.57	682.45	1367.92
37.50	179.83	3174.96	8756.47

PROOFS OF ADDITION.

76. To prove addition by varying the combinations.

RULE.—*Begin with the right hand or unit column, and add the terms in each column in an opposite direction from that in which they were first added; if the two results agree, the work is supposed to be right.*

77. To prove addition by casting out 9's.

The remainder arising from dividing any number by 9 is always the same as the remainder that arises from dividing the sum of all its digits by 9.

This arises from the fact that every 10, 100, or 1000 of a number, when divided by 9, has a remainder of 1. Thus, 9 into 10, once and 1 over; 9 into 100, 11 times and 1 over, etc.; 9 into 20, twice and 2 over; into 200, 2000, etc., 2 over; into 30, 300, 3000, etc., 3 over. Hence, if we divide by 9 separately the parts of a number, the remainders will be expressed by the digits of the number. Thus, if we divide 2345 by 9, first dividing the 2000, the remainder is 2; dividing 300, the remainder is 3; dividing 40, the remainder is 4; dividing 50, the remainder is 5. Total remainder is $2 + 3 + 4 + 5 = 14$; excess of 9's in $14 = 5$, which is the same as the excess of 9's in 2345. Hence, any number is equal to an exact number of 9's plus the sum of its digits.

1. Prove by excess of 9's that $34852 + 24784 + 72456 = 132092$.

OPERATION.

34852,	excess of 9's =	4
24784,	excess of 9's =	7
72456,	excess of 9's =	6
132092,		17,
exc. of 9's = 8.		exc. of 9's = 8.

SOLUTION. — Adding the digits in the first number, the excess of 9's is 4; in the second, 7; and in the third, 6. The sum of these excesses is 17 and the excess of 9's in 17 is 8. Dividing 132092 by 9, we find the remainder to be 8; hence the work is probably correct.

RULE. — I. *Find the excess of 9's in the sum of the digits in each addend; add these excesses, and find the excess of 9's in their sum. If it agrees with the excess of 9's in the answer, the answer is probably correct.*

NOTES. — 1. It is evident that the same result will be obtained by adding the digits in columns as in rows.

2. The method of proving addition by the excess of 9's, fails in the following cases: 1st, when the figures of the answers are misplaced; 2d, when the value of one figure is as much too great as that of another is too small.

78. To prove addition by casting out 11's.

When a number expressed by a digit in an odd place is divided by 11, the remainder is equal to that digit; and a number expressed by a digit in an even place, lacks that digit of being a multiple of 11. Therefore, if a number expressed by two figures is divided by 11, the remainder equals the digit in the odd place minus the digit in the even place.

Thus, in 4500, 500, represented by 5 in the third place $= (45 \times 11) + 5$. 4000, represented by 4 in the fourth place $= (364 \times 11) - 4$. Hence, $4500 = (45 + 364) \times 11 + (5 - 4)$, and 4500 divided by 11 has a remainder of $5 - 4$ or 1.

NOTE. — If the digit in the even place is greater than that in the odd place, it cannot be subtracted, so we add one 11 to it, and then proceed to subtract.

PRINCIPLE. — *Any number divided by* 11 *will have a remainder equal to the sum of the digits in the odd places minus the sum of those in the even places.*

From this principle, we deduce a proof of addition by casting out 11's, similar to the proof by casting out 9's.

1. Prove by excess of 11's that $34852 + 24784 + 72456 = 132092$.

OPERATION.

$$\begin{array}{ll} 34852, & 13 - 9 = 4 \\ 24784, & 13 - 12 = 1 \\ \underline{72456,} & 17 - 7 = \underline{10} \\ 132092, & \quad 15, \text{ exc.} = 4. \\ (5 + 11) - 12 = 4, & \\ \quad \text{exc.} = 4. & \end{array}$$

SOLUTION. — The sums of the digits in the odd places minus those in the even places in the various addends are $4 + 1 + 10 = 15$, and the excess of 11's in $15 = 4$. The sum of the digits in the odd places of 132092 minus those in the even places $= 4$; and the excess of 11's $= 4$. Hence, the answer is probably correct.

RULE. — *Subtract the sum of the digits in the even places from the sum of those in the odd places in each addend. Add the results and find the excess of* 11's *in their sum. If this agrees with the excess of* 11's *in the answer, the answer is probably correct.*

79. To prove addition by summing up the digits.

If we add the digits of any number consisting of two or more digits, we find that their sum is the same as the excess of 9's in the number. Thus, in 12 the sum of the digits is 3, and the excess of 9's is 3; in 14, the sum of the digits is 5, excess of 9's is 5; in 125, sum of digits is 8, excess of 9's is 8, etc. Hence, we may prove addition by adding the digits, and then *adding the digits in their sum*, instead of by casting out the 9's. The last digit obtained is called the **final digit**.

1. Prove by summing up the digits that $34852 + 24784 + 72456 = 132092$.

OPERATION.

$$\left.\begin{array}{r} 34852 \\ 24784 \\ \underline{72456} \end{array}\right\} \text{8, final digit.}$$
$$132092 = \text{8, final digit.}$$

SOLUTION. — Adding the first row $3 + 4 + 8 + 5 + 2 = 22$, and the sum of the digits in $22 = 4$. Adding the 4 to the second row, $4 + 2 + 4 + 7 + 8 + 4 = 29$, sum of digits 11; third row, $11 + 7 + 2 + 4 + 5 + 6 = 35$, sum of digits, 8 the final digit. In the sum 132092, $1 + 3 + 2 + 0 + 9 + 2 = 17$, and the sum of these digits is 8. Hence the answer is probably correct.

RULE.—*Add the digits in the first addend, and find the sum of the digits in the result. Add this sum to the sum of the digits in the second addend, and sum up as before. Proceed in this manner until the last addend is summed up and the final digit is found. Then sum up the digits in the answer to be proved. If the result is the same as the final digit of the addends, the answer is probably correct.*

Examples in the Preceding Rules.

80. 1. The area and population of the North Atlantic division of the United States in 1890 were as follows:

	AREA.	POPULATION.
Maine	33040	661086
New Hampshire	9305	376530
Vermont	9565	332422
Massachusetts	8315	2238943
Rhode Island	1250	345506
Connecticut	4990	746258
New York	49170	5997853
New Jersey	7815	1444933
Pennsylvania	45215	5258014

Find the total area and population of this division.

2. The area and population of the South Atlantic division in 1890 were as follows:

	AREA.	POPULATION.
Delaware	2050	168493
Maryland	12210	1042390
District of Columbia	70	230392
Virginia	42450	1655980
West Virginia	24780	762794
North Carolina	52250	1617947
South Carolina	30570	1151149
Georgia	59475	1837353
Florida	58680	391422

What was the total area of this division? The total population?

3. The area and population of the North Central division in 1890 were as follows:

	Area.	Population.
Ohio	41060	3672316
Indiana	36350	2192404
Illinois	56650	3826351
Michigan	58915	2093889
Wisconsin	56040	1686880
Minnesota	83365	1301826
Iowa	56025	1911896
Missouri	69415	2679184
North Dakota	70795	182719
South Dakota	77650	328808
Nebraska	77510	1058910
Kansas	82080	1427096

What was the total population of this division? The total area?

4. The area and population of the South Central division in 1890 were as follows:

	Area.	Population.
Kentucky	40400	1858635
Tennessee	42050	1767518
Alabama	52250	1513017
Mississippi	46810	1289600
Louisiana	48720	1118587
Texas	265780	2235523
Oklahoma	39030	61834
Arkansas	53850	1128179

Find the total population; the total area.

5. The Western division was estimated as follows in 1890:

	Area.	Population.
Montana	146080	132159
Wyoming	97890	60705
Colorado	103925	412198
New Mexico	122580	153593
Arizona	113020	59620
Utah	84970	207905
Nevada	110700	45761
Idaho	84800	84385
Washington	69180	349390
Oregon	96030	313767
California	158360	1208130

Find the total population of this division, and the total area.

6. In addition, the United States contained in 1890 the following areas and populations:

	AREA.	POPULATION.
Alaska	577390	32052
Indian Territory	31400	179321
Untaxed Indians		146143
Delaware, Raritan, and New York Bays,	720	
U. S. portion of Great Lakes	65177	

Find the total area and population of the United States.

7. A man bequeathed his estate as follows: to each of his two sons, $12450; to each of his three daughters, $6500; to his wife, $650 more than to both the sons; and the remainder, which was $1000 more than he had left to all his family, he gave to benevolent institutions. What was the whole amount of his property?

8. Iron was discovered in Greece by the burning of Mount Ida, B.C. 1406; and the electro-magnetic telegraph was invented by Morse, A.D. 1832. What period of time elapsed between the two events?

9. The population of the ten largest cities of the United States by the census of 1890 was as follows:

New York, N.Y. . . .	1515301	Boston, Mass.	448477
Chicago, Ill.	1099850	Baltimore, Md.	434439
Philadelphia, Pa. . . .	1046964	San Francisco, Cal. . .	298997
Brooklyn, N.Y. . . .	806343	Cincinnati, O.	296908
St. Louis, Mo. . . .	451770	Cleveland, O.	261353

What was the entire population of these cities?

10. A farm has five corners: from the first to the second it is 34.72 rods; from the second to the third, 48.44 rods; from the third to the fourth, 152.17 rods; from the fourth to the fifth, 95.36 rods; and from the fifth to the first, 56.18 rods. What is the whole distance around the farm?

11. The number of people speaking the English language is estimated to be 111100000; the number speaking French, 51200000; German, 75200000; Italian, 33400000; Spanish, 26190000. Find the total number of people speaking these five languages.

SUBTRACTION.

81. **Subtraction** is the process of taking away part of a number, or of determining the difference between two numbers of the same unit value.

82. The **Minuend** is the number to be subtracted from.

83. The **Subtrahend** is the number to subtract.

84. The **Remainder** or **Difference** is the result obtained by the subtraction.

85. When the given numbers contain more than one figure each, the method of subtraction depends upon the following principles:

PRINCIPLES. — I. *If the units of each order in the subtrahend are taken separately from the units of like order in the minuend, the sum of the differences must be equal to the entire difference of the given numbers.* (Ax. 10.)

II. *If the minuend and subtrahend are equally increased, the remainder will not be changed.*

Examples.

86. 1. From 928 take 275.

OPERATION.

Minuend,	928
Subtrahend,	275
Remainder,	653

SOLUTION. — We first subtract 5 units from 8 units, and obtain 3 units for a partial remainder. As we cannot take 7 tens from 2 tens, we add 10 tens to the 2 tens, making 12 tens; then 7 tens from 12 tens leave 5 tens, the second partial remainder. Now, since we added 10 tens, or 1 hundred, to the minuend, if we add 1 hundred to the subtrahend, the true remainder will not be changed (§ 85, II); thus, 1 hundred

added to 2 hundreds makes 3 hundreds, and this sum subtracted from 9 hundreds leaves 6 hundreds. Hence we have for the remainder, 653.

NOTE. — The process of adding 10 to the minuend is sometimes called *borrowing* 10, and that of adding 1 to the next figure of the subtrahend, *carrying* 1.

The following method is often preferred:

OPERATION.

	(8)	(12)	
Minuend,	9	2	8
Subtrahend,	2	7	5
Remainder,	6	5	3

SOLUTION. — We subtract 5 units from 8 units, and 3 units remain. Since we cannot take 7 tens from 2 tens, we take 1 hundred from the 9 hundreds and reduce it to tens; 1 hundred = 10 tens; 10 tens and 2 tens = 12 tens; 7 tens from 12 tens leave 5 tens. Since we have taken 1 hundred from the 9 hundreds, only 8 hundreds are left; 2 hundreds from 8 hundreds leave 6 hundreds. Hence we have for the remainder 653.

NOTE. — The numbers written over the minuend are used simply to explain more clearly the method of subtracting; in practice the process should be performed mentally, and these numbers should be omitted.

2. From 929.635 take 537.3129.

OPERATION.

929.635
537.3129
392.3221 Rem.

SOLUTION. — We place the subtrahend under the minuend so that units of the same order stand in the same column, and then proceed as in whole numbers, placing the decimal point in the remainder directly below those in the minuend and subtrahend, that is, between tenths and units.

RULE. — I. *Write the subtrahend under the minuend, placing terms of the same order under each other.*

II. *Begin at the units, and take each term of the subtrahend from the corresponding term of the minuend, writing the remainder underneath.*

III. *If any term in the subtrahend is greater than the corresponding term in the minuend, add* 10 *to the latter and subtract. Then add* 1 *to the next term of the subtrahend (or subtract* 1 *from the next term of the minuend) and proceed as before.*

IV. *If there are any decimal orders in the minuend or subtrahend, place the decimal point in the remainder directly below those above it.*

	3.	4.	5.	6.
From	47965	103767	57610218	89764.321
Take	26714	98731	8306429	83720.595

Find the remainders:

7. 58000 − 212.	14. 180037561 − 5703746.
8. 4050 − 389.	15. 2460371219 − 98720342.
9. 22012 − .05.	16. 890374261 − 24350367.
10. 3456.1 − 185.	17. 10000030 − 999999.
11. 1.0066 − .15.	18. 222222 − 8.888.
12. 1000 − .001.	19. 15.6056 − 3.2.
13. 10000 − .0001.	20. 9.8765431 − .009085.

87. To subtract two or more numbers at once.

1. A man having 1278 barrels of flour, sold 236 barrels to A, 362 to B, and 387 to C; how many had he left?

OPERATION.

Minuend,	1278
Subtrahends,	236
	362
	387
Remainder,	293

SOLUTION.—Since the remainder sought, added to the subtrahends, must be equal to the minuend, we add the columns of the subtrahends, and supply such numbers in the remainder as, combined with these sums, will produce the minuend. Thus, 7 and 2 are 9, and 6 are 15, and 3 (supplied in the remainder sought) are 18; then, carrying the tens' term of the 18, 1 and 8 are 9, and 6 are 15, and 3 are 18, and 9 (supplied in the remainder) are 27; lastly, 2 to carry to 3 are 5, and 3 are 8, and 2 are 10, and 2 (supplied in the remainder) are 12; and the whole remainder is 293.

RULE.—I. *Write the several subtrahends under the minuend, add the first column of the subtrahends, and supply such a number in the remainder sought, as, added to this partial sum, will give an amount having for its unit term the term above in the minuend.*

II. *Carry the tens' term of this amount to the next column of the subtrahends, and proceed as before till the entire remainder is obtained.*

	2.	3.	4.	5.
From	47962	127362	903486	2503734
Take	21435	56304	430164	89763
	15672	4782	132875	94207
	456	9156	67321	237564

88. Short method when the minuend consists of one or more digits of any order higher than the highest order in the subtrahend.

The difference between any number and a unit of the next higher order is called an **Arithmetical Complement.** Thus, 4 is the arithmetical complement of 6, 31 of 69, 2792 of 7208, etc.

1. Subtract 29876 from 3400000.

OPERATION.

$$\begin{array}{r} 3400000 \\ 29876 \\ \hline 3370124 \end{array}\ \textit{Ans.}$$

SOLUTION.—To subtract 29876 from 3400000 is the same as to subtract a number one less than 29876, or 29875, from 3399999 (Ax. 2). We therefore diminish the 34 of the minuend by 1, and then take each term of the subtrahend from 9, except the lowest or right-hand term, which we subtract from 10.

RULE.—I. *Subtract* 1 *from the significant part of the minuend and write the remainder, if any, as a part of the result.*

II. *Proceeding to the right, subtract each term in the subtrahend from* 9, *except the last, which subtract from* 10.

Subtract:

2. 756 from 1000.
3. 8576 from 4000000.
4. 5768 from 10000.
5. 6981 from 100000.
6. 13057 from 1700000.
7. 90.59876 from 64000.
8. 599948 from 1000000.
9. 8431.5 from 20000.

10. What is the arithmetical complement of 271? of 18365? of 3401250?

89. A **Debtor**, in business transactions, is a purchaser, or a person who receives money, goods, or services from another; and a **Creditor** is a seller, or a person who parts with money, goods, or services to another.

90. Business accounts have two sides: the Dr. or Debit side, on which are recorded the sums a person owes, and the Cr. or Credit side, which contains the sums owing to him. The difference between the two sides is called the **Balance of the Account.**

91. Find the balance of the following accounts:

1.

Dr. ANDREW CLARKE. *Cr.*

1895			1895		
May 3	*To Mdse.*	*$157.65*	*May 6*	*By Cash*	*$ 75.00*
" 9	*" Sundries*	*65.09*	*" 10*	*" Goods ret'd*	*65.09*
" 15	*" Cash*	*25.00*	*" 17*	*" Note*	*200.00*
" 21	*" Mdse.*	*175.73*	*" 25*	*" Note*	*300.00*
" 29	*" "*	*610.25*		*" Balance*	
June 1	*To Balance*				

2.

Dr. JOSEPH SEWALL. *Cr.*

1896			1896		
Jan. 1	*To Sundries*	*$1342.95*	*Jan. 6*	*By Check*	*$500.00*
" 5	*" Mdse.*	*198.16*	*" 18*	*" Note*	*350.00*
" 17	*" Cash*	*15.05*	*" 20*	*" Check*	*725.00*
" 31	*" Mdse.*	*702.21*		*" Balance*	
Feb. 1	*To Balance*				

PROOFS OF SUBTRACTION.

92. To prove subtraction by adding the subtrahend and remainder, etc.

Since the minuend *minus* the subtrahend equals the remainder, the remainder *plus* the subtrahend must equal the minuend, and the minuend *minus* the remainder must equal the subtrahend.

RULE. — I. *Add the remainder to the subtrahend. If the result equals the minuend, the answer is correct.*

II. *Subtract the remainder from the minuend. If the result equals the subtrahend, the answer is correct.*

93. To prove subtraction by casting out 9's.

1. Prove by excess of 9's that 15964 – 9432 = 6532.

OPERATION.

15964, excess of 9's = 7
9432, excess of 9's = 0
6532, exc. of 9's = 7. 7, exc. of 9's.

SOLUTION.—Casting out the 9's from the minuend (§ 77), the remainder is 7, and casting out the 9's from the subtrahend, the remainder is 0. The difference, 7, is the same as the excess of 9's in 6532; hence this answer is probably correct.

RULE.—*Subtract the excess of* 9*'s in the subtrahend from that in the minuend, casting out the* 9*'s from the remainder. Compare the excess with the excess in the answer.*

NOTE.—If the excess of 9's in the subtrahend is greater than that in the minuend, add one 9 to the minuend and then subtract.

94. To prove subtraction by casting out 11's.

1. Prove by excess of 11's that 15964 – 9432 = 6532.

OPERATION.

15964, 14 – 11 = 3 (+ 11)
9432, 6 (+ 11) – 12 = 5
6532, 9, exc.
7 (+ 11) – 9 = 9, exc.

SOLUTION.—In the minuend the sum of the digits in the odd places is 14, and in the even places 11 (§ 78). Their difference is 3. In the subtrahend, the sum of the digits in the odd places is 6, and in the even places 12. Since 12 cannot be subtracted from 6, we add one 11 to 6, making 17, and 17 – 12 = 5 We subtract the results, but since 5 cannot be subtracted from 3, we add one 11 to the 3, making 14; 14 – 5 = 9, the excess of 11's. In the same way, we find the excess of 11's in 6532 to be 9. Since the two excesses of 11 agree, the answer is probably correct.

RULE.—*Subtract the excess of* 11*'s in the subtrahend from that in the minuend, casting out the* 11*'s from the remainder. Compare the excess with the excess in the answer.*

95. To prove subtraction by summing up the digits.

1. Prove by summing up the digits that 15964 – 9432 = 6532.

OPERATION.

15964, 7, final digit.
9432 }
6532 } 7, final digit.

SOLUTION.—Summing the digits in the subtrahend and remainder, we have 9 + 4 + 3 + 2 = 18; sum of digits 9; 9 + 6 + 5 + 3 + 2 = 25; sum of digits 7, the *final digit* (§ 79). Summing up the digits in the minuend, 1 + 5 + 9 + 6 + 4 = 25; final digit 7. Hence the answer is probably correct.

RULE.—*Sum up the digits in the subtrahend and remainder and find the final digit. Compare this with the final digit found by summing up the digits in the minuend.*

Examples in Preceding Rules.

96. 1. The population of New York City in 1880 was 1206299; in 1890 it was 1515301. What was the increase in the ten years?

2. The first newspaper published in America was issued in Boston in 1704. How long was that before the death of Benjamin Franklin, which occurred in 1790?

3. A merchant sold a quantity of goods for $ 42017.75, which was $ 1675.36 more than they cost him. How much did they cost him?

4. A speculator having 57436 acres of land sold at different times 53.75 acres and 1765.25 acres. How much land has he remaining?

5. The exports of the United States for the year ending June 30, 1894, amounted to $ 972861378, and the imports to $ 740730293. How much did the exports exceed the imports?

6. For the year ending June 30, 1894, the total coinage in the United States was $ 106216730.06; the gold amounted to $ 99474912.50, and the bronze and nickel to $ 716919.26. What was the value of the silver coinage?

7. The mineral products of the United States in 1892 amounted to $ 688687712; in 1893 to $ 609817495. What was the decrease in value?

8. The area of the Chinese Empire in 1890 was 4291391 square miles, and that of the United States 3668167 square miles; the estimated population of the former was 361500000, and that of the latter was 62979766. What was the difference in area and in population?

9. The population of New York in 1890 was 1515301, and that of Boston 448477. How many more inhabitants had New York than Boston?

10. The total length of railroads in operation in the United States in 1859 was 27857 miles; in 1893 it was 173433 miles. What was the increase from 1859 to 1893?

11. The South Atlantic division of the United States in 1890 had a population of 8857920, the Western division 3027613, and the North Atlantic division 17401545. How many more inhabitants had the last named division than the other two?

12. Having \$20000, I wish to know how much more I must accumulate to be able to purchase a piece of property worth \$23470, and have \$5400 left?

13. A has \$3540.75 more than B, and \$1200.63 less than C, who has \$20600; D has as much as A and B together. How much has D?

14. The revenue of the United States Post Office in 1873 was \$22996742; in 1883, \$45508693; in 1893, \$75896933. How much did it increase from 1873 to 1883? From 1883 to 1893? From 1873 to 1893?

15. From 4568 take 1320.12 + 275.39 + 320.35.

16. Subtract 1200.9 + 750.6 + 96.3 from 4756 + 575 + 140 + 84.9.

17. A man bought four city lots, for which he paid \$15760. For the first he paid \$2175, for the second \$3794, and for the third \$4587. How much did he pay for the fourth?

18. John Wise owns property to the amount of \$75860, of which he has \$45640 invested in real estate, \$25175 in personal property, and the remainder he has in bank. How much has he in bank?

19. Lake Huron contains 23800 square miles. By how much does it exceed the area of Lake Erie and Lake Ontario, the former containing 9960 square miles, and the latter 7240 square miles?

20. In the years 1881 to 1890, there arrived in the United States 5246613 immigrants, of whom 655381 came from Ireland, and 1452952 from Germany. How many came from other countries?

21. A speculator gained $ 5760, and afterward lost $ 2746; at another time he gained $ 3575, and then lost $ 4632. How much did his gains exceed his losses?

22. The North Atlantic division of the United States has an area of 168665 square miles, the South Atlantic division, 282535 square miles, and the Western division, 1187535 square miles. How much greater is the area of the Western division than of the North and South Atlantic divisions combined?

23. The total expenditure of the United States Post Office Department for 1893 was $ 81074104, of which sum $ 41170054 was paid for the transportation of the mail, and $ 15862621 as compensation to postmasters. How much was expended for all other purposes?

24. The gold coinage of the United States Mint for the year ending June 30, 1894, was as follows: double eagles, $ 55143640; eagles, $ 34968840; half eagles, $ 9287180; quarter eagles, $ 75252.50. The silver coinage for the same year was: standard dollars, $ 758; half dollars, $ 3363327; quarter dollars, $ 2296595; dimes, $ 364218.30. How much did the gold coinage exceed the silver in value?

25. The population of the United States in 1890 was as follows: total, 62622250; North Atlantic division, 17401545; South Atlantic division, 8857920; North Central division, 22362279; South Central division, 10972893. What was the population of the Western division?

26. What number must be added to 32456.0927 to make 56534?

27. If I subtract 15213.05 from 325020.1, and then add 36211, how much more must I add to make 1000000?

28. A business man had on deposit in a bank $ 1250, and on Monday he deposited $ 350 more. Tuesday he drew out by check $ 1177, Wednesday morning deposited $ 2677.87, and drew out by check in the afternoon $ 165.72. Thursday he deposited $ 3247, and Saturday drew out $ 5640 to pay salaries and bills. What amount was still on deposit in the bank?

MULTIPLICATION.

97. Multiplication is the process of taking one of two given numbers as many times as there are units in the other.

98. The **Multiplicand** is the number to be multiplied.

99. The **Multiplier** is the number by which to multiply, and shows how many times the multiplicand is to be taken.

100. The **Product** is the result obtained by the process of multiplication.

101. The **Factors** are the multiplicand and multiplier.

NOTES.—1. Factors are producers, and the multiplicand and multiplier are called factors because they produce the product.

2. Multiplication is a short method of performing addition when the numbers to be added are equal.

3. The *multiplier* is always *regarded* as an *abstract number;* but either factor of a product may be *used* abstractly, even if it is concrete; hence, either factor may be used as a multiplier.

4. The true multiplicand, however, is the factor that would be used as one of the equal parts in addition.

102. It is evident that 5 units taken 3 times is the same as 3 units taken 5 times; and the same is true of any two factors. The method of multiplying when either factor contains more than one figure, depends upon the following principles:

PRINCIPLES.—I. *The product of any two factors is the same, whichever is used as the multiplier. If units are multiplied by units, the product will be units; if tens are multiplied by units, or units by tens, the product will be tens; and so on.*

II. *If either factor is units of the first order, the product will be units of the same order as the other factor.*

III. *If the units of each order in the multiplicand are taken separately as many times as there are units in the multiplier, the*

sum of the products must be equal to the entire product of the given numbers. (Ax. 10).

MULTIPLICATION TABLE.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32	34	36	38	40
3	6	9	12	15	18	21	24	27	30	33	36	39	42	45	48	51	54	57	60
4	8	12	16	20	24	28	32	36	40	44	48	52	56	60	64	68	72	76	80
5	10	15	20	25	30	35	40	45	50	55	60	65	70	75	80	85	90	95	100
6	12	18	24	30	36	42	48	54	60	66	72	78	84	90	96	102	108	114	120
7	14	21	28	35	42	49	56	63	70	77	84	91	98	105	112	119	126	133	140
8	16	24	32	40	48	56	64	72	80	88	96	104	112	120	128	136	144	152	160
9	18	27	36	45	54	63	72	81	90	99	108	117	126	135	144	153	162	171	180
10	20	30	40	50	60	70	80	90	100	110	120	130	140	150	160	170	180	190	200
11	22	33	44	55	66	77	88	99	110	121	132	143	154	165	176	187	198	209	220
12	24	36	48	60	72	84	96	108	120	132	144	156	168	180	192	204	216	228	240
13	26	39	52	65	78	91	104	117	130	143	156	169	182	195	208	221	234	247	260
14	28	42	56	70	84	98	112	126	140	154	168	182	196	210	224	238	252	266	280
15	30	45	60	75	90	105	120	135	150	165	180	195	210	225	240	255	270	285	300
16	32	48	64	80	96	112	128	144	160	176	192	208	224	240	256	272	288	304	320
17	34	51	68	85	102	119	136	153	170	187	204	221	238	255	272	289	306	323	340
18	36	54	72	90	108	126	144	162	180	198	216	234	252	270	288	306	324	342	360
19	38	57	76	95	114	133	152	171	190	209	228	247	266	285	304	323	342	361	380
20	40	60	80	100	120	140	160	180	200	220	240	260	280	300	320	340	360	380	400

Examples.

103. 1. Multiply 346 by 8.

OPERATION.

Multiplicand,	346
Multiplier,	8
Product,	2768

SOLUTION. — In this example it is required to take 346 eight times. If we take the units of each order 8 times, we shall take the entire number 8 times (§ 102, III). Therefore, commencing at the right hand, we say: 8 times 6 units are 48 units, or 4 tens and 8 units; writing the 8 units in the product in units' place, we reserve the 4 tens to add to the next product; 8 times 4 tens are 32 tens, and the 4 tens reserved in the last product added are 36 tens, or 3 hundreds and 6 tens; we write the 6 tens in the product in tens' place, and reserve the 3 hundreds to add to the next product; 8 times 3 hundreds are 24 hundreds, and the 3 hundreds reserved in the last product added are 27 hundreds, which being written in the product, with each figure in the place of its order, gives for the entire product, 2768.

2. Multiply 34.6 by 8.

OPERATION.

```
  34.6
     8
 -----
 276.8 Ans.
```

SOLUTION. — Proceeding as in Ex. 1, we find that 8 times 6 tenths are 48 tenths, or 4 units and 8 tenths. We write the 8 tenths in the tenths' place in the product, and reserve the 4 units to add to the next product; 8 times 4 units are 32 units, and the 4 units reserved are 36 units, or 3 tens and 6 units. We write the 6 in units' place, and reserve the 3 tens; 8 times 3 tens are 24 tens and 3 tens reserved are 27 tens; hence the entire product is 276.8.

3. Multiply 758 by 346.

OPERATION.

```
    758
    346
 ------
   4548
  3032
 2274
 ------
 262268 Ans.
```

SOLUTION. — In this example the multiplicand is to be taken 346 times, which may be done by taking the multiplicand separately as many times as there are units expressed by each figure of the multiplier. 758 multiplied by 6 units is 4548 *units* (§ 102, II); 758 multiplied by 4 tens is 3032 *tens* (§ 102, II), which we write with its lowest order in tens' place, or under the figure used as a multiplier; 758 multiplied by 3 hundreds is 2274 *hundreds* (§ 102, II), which we write with its lowest order in hundreds' place. Since the sum of these products must be the entire product of the given numbers (§ 102, III), we add the results, and obtain the product, 262268.

NOTES. — 1. When the multiplier contains two or more terms, the several results obtained by multiplying by each term are called *partial products.*

2. When there are ciphers between the significant terms of the multiplicand, pass over them, and multiply by the significant terms only.

RULE. — I. *Write the multiplier under the multiplicand, placing units of the same order under each other.*

II. *Multiply the multiplicand by each term of the multiplier successively, beginning with the right-hand term, and write the first term of each partial product under the term of the multiplier used, writing down and carrying as in addition.*

III. *Add the partial products. If the multiplicand contains decimal places, point off the same number in the product, and the result will be the true product.*

NOTE. — The method of pointing off decimal places in the product when both multiplicand and multiplier contain decimal places will be explained in § 236.

	4.	5.	6.	7.
Multiply	475	3172	9827	7198
By	9	14	84	216

Multiply:

8. 314.16 by 175.
9. 40930 by 779.
10. 46481 by 936.
11. 1560.7 by 3094.
12. 281216 by 978.
13. 302.04 by 4267.

14. What will be the cost of building 276 miles of railroad at $ 61320 per mile?

15. If 125 tons of iron rail are required for one mile of railroad, how many tons will be required for 196 miles?

16. A tailor bought 36 pieces of broadcloth, each piece containing 47 yards, at $ 7 a yard. How much did he pay for the whole?

17. The salary of a member of Congress is $ 5000 a year, and in the Fifty-fourth Congress, ending March 4, 1897, there were 356 members. How much did they all receive?

18. At a profit of $ .15 on a dollar, how much money would I make on goods costing $ 759?

19. A teacher receives a discount on books equal to .25 of their value. What will be his bill for two books listed at $ 1.00 and $ 2.00 respectively?

NOTE. — *Per cent* is another name for hundredths and its sign is %. Instead of saying .25 we might say 25 *per cent* or 25 %.

20. At the rate of $ .06 for every dollar, how much interest should be paid on $ 500 loaned for 1 year? How much should be paid for the same sum loaned for 5 years?

21. One pound (£) sterling is worth $ 4.8665. If I wish to buy £225 English money, how much United States money must I give in exchange?

22. The United States mints, in the year ending June, 1894, coined 2757182 double eagles ($ 20), 3496884 eagles ($ 10), 1857436 half eagles ($ 5), 30101 quarter eagles ($ 2.50), 758 standard dollars, 6726654 half dollars, 9186380 quarter dollars, 3642183 dimes, 9226071 five-cent pieces, 25561571 one-cent pieces. What was the total value of that year's coinage?

23. A man having £2000 wishes to exchange it for United States money. How much will he receive?

24. What must be paid for insuring a house worth $ 9000, if the rate charged by the insurance company is $ .02 on $ 1?

25. If I buy 65 shares of a stock at $ 86 a share, and sell 35 shares at $ 80 a share, and 30 shares at $ 110 a share, do I gain or lose, and how much?

26. Mr. Armstrong bought a house for $ 15760, and gained 25% of its cost by selling it. For what price did he sell it?

27. At $ 110 a share, what will be the cost of 50 shares of railroad stock? If a broker charges me $ .00125 on a dollar for buying this stock, how much must I pay him for his services?

28. The circumference of every circle is 3.1416 times the diameter. Find the circumference of a circle whose diameter is 26 inches.

POWERS OF NUMBERS.

104. A **Power** is the product arising from multiplying a number by itself or using it a certain number of times as a factor. Thus, 8 is a power of 2, because $2 \times 2 \times 2 = 8$.

105. A **Root** is a factor repeated to produce a power. Thus, 2 is a root of 8.

106. An **Index** or **Exponent** is a figure indicating the power to which a number is to be raised. Thus, $2 \times 2 \times 2$ may be indicated by 2^3, and the small figure 3 placed above and to the right of the 2 is the index or exponent.

107. The **First Power** of any number is the number itself, or the root. Thus, 2, 3, 5, are first powers or roots.

108. The **Second Power,** or **Square,** of a number is the product arising from using the number two times as a factor. Thus, $2^2 = 2 \times 2 = 4$; $5^2 = 5 \times 5 = 25$.

109. The **Third Power,** or **Cube,** of a number is the product arising from using the number three times as a factor. Thus, $4^3 = 4 \times 4 \times 4 = 64$.

NOTE.—The higher powers are named in their order as *fourth power*, *fifth power*, *sixth power*, etc.

110. The process of producing any required power by multiplication is called **Involution.**

Examples.

111. To find any power of a number.

1. What is the third power or cube of 23?

OPERATION.

$23 \times 23 \times 23 = 12167$ *Ans.*

SOLUTION. — We multiply 23 by 23, and the product by 23; and, since 23 has been taken 3 times as a factor, the last product, 12167, must be the third power or cube of 23.

RULE. — *Multiply the number by itself as many times, less* 1, *as there are units in the exponent of the required power.*

Find:

2. The square of 72.
3. The fifth power of 12.
4. The cube of 25.
5. The seventh power of 7.
6. The fourth power of 19.
7. The sixth power of 3.

Find the powers indicated by the following expressions:

8. 9^5.
9. 11^3.
10. 18^2.
11. 125^4.
12. 786^2.
13. 94^6.
14. 100^4.
15. 17^3.
16. 251^5.
17. 110^6.
18. Multiply 8^3 by 15^2.
19. Multiply 25^2 by 3.

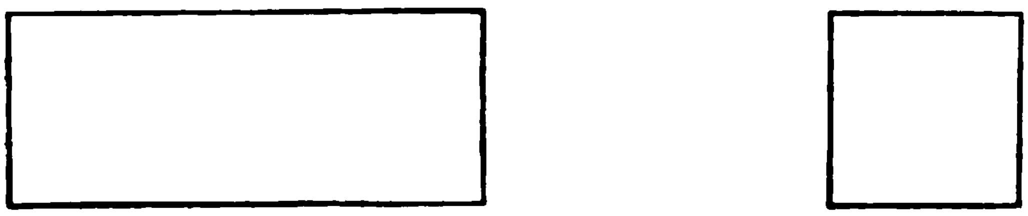
SURFACES AND VOLUMES.

112. A **Surface** is the bounding or limiting part of a body. Every surface has two dimensions, — length and breadth.

NOTE. — Surfaces are either *plane* (flat) or *curved.*

113. The **Area** of a surface is its contents reckoned in square units.

114. A **Rectangle** is a four-sided plane figure having its



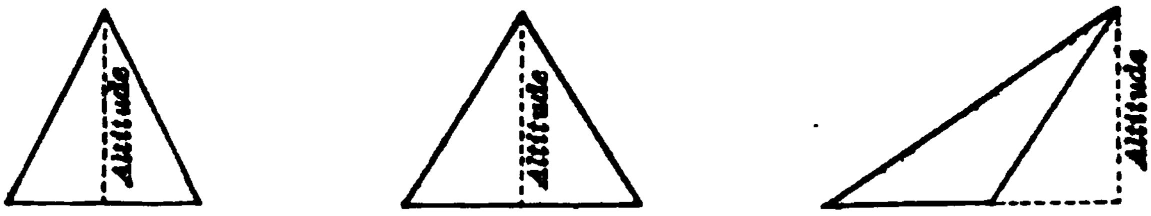
RECTANGLES.

opposite sides parallel and whose angles are all right angles, that is, angles of 90°.

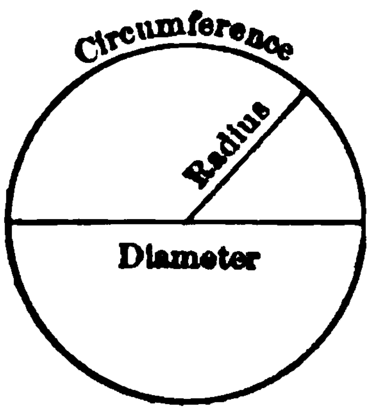
115. A **Triangle** is a plane figure bounded by three sides and having three angles.

Its **Base** is the side on which it is supposed to stand.

Its **Altitude** is a line drawn from the angle opposite the base at right angles to the base.



TRIANGLES.



CIRCLE.

116. A **Circle** is a plane figure bounded by a curved line called the **Circumference,** every point of which is equally distant from a point within called the **Center.**

The **Diameter** is a line passing through the center to the circumference on both sides.

The **Radius** is a line from the center to any point in the circumference and is equal to one half the diameter.

NOTE.—For other plane surfaces see pages 470–488.

117. RULES.—I. *To find the area of a rectangle, multiply the number of units in length by the number of like units in width. The product will be the number of square units in area.*

II. *To find the area of a triangle, multiply the base by one half the altitude.*

III. *To find the area of a circle, multiply the square of the diameter by* .7854.

IV. *To find the circumference of a circle, multiply the diameter by* 3.1416.

118. A **Solid** is a figure or a portion of space that has three dimensions,—length, breadth, and thickness (height or depth).

119. The **Volume** of a solid is its contents reckoned in cubic units.

120. Rule. — *To find the volume of any rectangular solid, multiply the number of units in the length by the number of like units in the breadth, and this product by the number of like units in the thickness; that is, find the product of the length, breadth, and thickness.*

Examples.

121. 1. How many square yards of carpet will be required to cover a floor that is 5 yards long and 3 yards wide, and how much will it cost at $1.75 a yard?

2. My garden has two grass plats; one is 30 feet square, and the other is 20 feet long and 40 feet wide. Which is the larger, and how many more square feet does it contain?

Find the areas of the following triangles:

3. Base 7 feet, altitude 25 feet.

4. Base 25 feet, altitude 7 feet.

5. Altitude 36 feet, base 28.5 feet.

6. What will be the cost of a triangular piece of land whose base is 15 rods and altitude 10 rods at $65.75 a rod?

Find the circumference of the following circles:

7. Diameter 10 inches.

8. Diameter 3 yards.

9. Diameter 26 fect.

10. Diameter 201 yards.

11. What is the circumference of a wheel 5 feet in diameter?

Find the areas of the following circles:

12. Diameter 16 inches.

13. Diameter 364 feet.

14. Diameter 250 inches.

15. Diameter 375 feet.

16. What is the area of a circular flower bed 23 feet in diameter?

17. How many cubic feet of wood are there in a column of wood which is 25 feet long, 15 feet wide, and 10 feet thick?

18. How many cubic feet of earth must be removed in digging an excavation which is 275 feet long, 315 feet wide, and 108 feet deep?

19. Find the contents in cubic inches of a box 12.5 inches long, 14 inches wide, and 7 inches high.

GENERAL PRINCIPLES OF MULTIPLICATION.

122. There are certain general principles of multiplication, of use in various contractions and applications. These relate, 1st, to changing the factors by addition or subtraction; 2d, to the use of successive factors in continued multiplication.

123. Changing the terms by addition or subtraction.

The product equals either factor taken as many times as there are units in the other factor (§ 102, I). Hence we have the following principles:

PRINCIPLES.—I. *Adding* 1 *to either factor, adds the other factor to the product.*

II. *Subtracting* 1 *from either factor, subtracts the other factor from the product.* Hence,

III. ADDING *any number to either factor,* INCREASES *the product by as many times the other factor as there are units in the number added; and* SUBTRACTING *any number from either factor,* DIMINISHES *the product by as many times the other factor as there are units in the number subtracted.*

124. A **Continued Multiplication** is the process of finding the product of three or more factors, by multiplying the first by the second, this result by the third, and so on.

If any number, as 17, is multiplied by any other number, as 3, the result will be 3 times 17; if this result is multiplied by another number, as 5, the new product will be 5 times 3 times 17, which is evidently 15 times 17. Hence, $17 \times 3 \times 5 = 17 \times 15$, etc.

Since 5 times $3 = 3$ times 5 (§ 102, I), 17 multiplied by 5 times $3 = 17$ multiplied by 3 times 5; or $17 \times 3 \times 5 = 17 \times 5 \times 3$. Hence, the product is not changed by changing the orders of the factors.

125. These principles may be stated as follows:

PRINCIPLES.—I. *If a given number is multiplied by several factors in continued multiplication, the result will be the same as if the given number were multiplied by the product of the several multipliers.*

II. *The product of several factors in continued multiplication will be the same, in whatever order the factors are taken.*

CONTRACTIONS IN MULTIPLICATION.

126. A **Composite Number** is one that may be produced by multiplying together two or more numbers. Thus, 18 is a composite number, since $6 \times 3 = 18$; or, $9 \times 2 = 18$; or, $3 \times 3 \times 2 = 18$.

127. The **Component Factors** of a number are the several numbers whose product is the given number. Thus, the component factors of 20 are 10 and 2, ($10 \times 2 = 20$); or, 4 and 5, ($4 \times 5 = 20$); or, 2 and 2 and 5, ($2 \times 2 \times 5 = 20$).

NOTE.—The pupil must not confound the *factors* with the *parts* of a number. Thus, the *factors* of which 12 is composed are 4 and 3, ($4 \times 3 = 12$); while the *parts* of which 12 is composed are 8 and 4, ($8 + 4 = 12$); or, 10 and 2, ($10 + 2 = 12$), etc. The *factors* are *multiplied*, while the *parts* are *added*, to produce the number.

Examples.

128. When the multiplier is a composite number.

1. Multiply 327 by 35.

OPERATION.

$$\begin{array}{r} 327 \\ 7 \\ \hline 2289 \\ 5 \\ \hline 11445 \end{array} \text{ Ans.}$$

SOLUTION.—The factors of 35 are 7 and 5. We multiply 327 by 7, and this result by 5, and obtain 11445, which must be the same as the product of 327 by 35 (§ 125, I).

RULE.—I. *Separate the composite number into two or more factors.*

II. *Multiply the multiplicand by one of these factors, and the product by another, and so on until all the factors have been used successively; the last product will be the product required.*

NOTE.—The factors may be used in any order that is most convenient (§ 125, II).

Find the products:

2. 736×24.

3. 538×56.

4. 27865×84.

5. 7856×144.

6. If a river discharges 17740872.5 cubic feet of water in one hour, how much will it discharge in 96 hours?

129. When the multiplier is 11.

1. Multiply 16439 by 11.

OPERATION.

```
16439      16439
   11         11
-----      ------
16439      180829 Ans.
16439
------
180829
```

SOLUTION. — If we multiply by the regular rule we notice that the second partial product is the same as the first, except that each term is one place further to the left; and we note that the total product is formed by adding each term of the multiplicand to the one preceding. Thus, $(0 + 9 = 9)$ and we write 9 for the units; $(9 + 3 = 12)$, we write 2; $(3 + 4 = 7$ and 1 carried $= 8)$, we write 8; $(4 + 6 = 10)$, we write 0; $(6 + 1 = 7$ and $1 = 8)$, we write 8; $(1 + 0 = 1)$, we write 1, and the answer is 180829.

RULE. — *Beginning with units, add each term of the multiplicand to the one preceding, carrying as in the regular rule.*

Multiply by 11:

2. 1495.	4. 52129.	6. 101209.
3. 1876.	5. 69872.	7. 315684.

130. When the multiplier is a unit of any order.

1. Multiply 365 by 1000.

OPERATION.

$365 \times 1000 = 365000$ *Ans.*

SOLUTION. — If we annex a cipher to the multiplicand, each term is removed *one* place toward the left, and consequently the value of the whole number is increased tenfold (§ 52, III). If two ciphers are annexed, each term is removed *two* places toward the left, and the value of the number is increased one hundredfold; and every additional cipher increases the value tenfold.

RULE. — *Annex as many ciphers to the multiplicand as there are ciphers in the multiplier.*

Multiply:

2. 364 by 100.	5. 16020 by 10000.
3. 248 by 1000.	6. 23999 by 100000.
4. 22913 by 10000.	7. 2056 by 1000000.

8. What is the cost of 1000 head of cattle at $ 50 each?

9. Multiply one million by one hundred thousand.

10. How many letters will there be on 100 sheets, if each sheet has 100 lines, and each line 100 letters?

131. When there are ciphers at the right hand of one or both of the factors.

1. Multiply 7200 by 40.

OPERATION.
72
40
288000 *Ans.*

SOLUTION.—The multiplicand, factored, is equal to 72×100; the multiplier, factored, is equal to 4×10; and as these factors taken in any order will give the same product (§ 125, II), we first multiply 72 by 4, then this product by 100 by annexing two ciphers, and this product by ten by annexing one cipher.

RULE.—*Multiply the significant terms of the multiplicand by those of the multiplier, and to the product annex as many ciphers as there are ciphers on the right of both factors.*

Find the products of:

2. 740×300.
3. 36000×240.
4. 20700×500.
5. 4007000×300.2.
6. 300200×640.
7. 2510000×3.56.

8. At the rate of \$5 on \$1000, what will be the tax on a piece of property valued at \$56000?

9. A salesman sells 5000 pieces of goods, each containing 40 yards, at \$1.50 a yard. What is the amount of his sales?

10. A dealer sold 300 horses at an average price of \$200 each. How much did he receive for them?

11. A certain state, having an area of 41000 square miles, has an average of 90 inhabitants to the square mile. What is the population of the state?

132. When one part of the multiplier is a factor of another part.

1. Multiply 4739 by 357.

OPERATION.	
4739	
357	
33173	Prod. by 7 units.
165865	Prod. by 35 tens.
1691823 *Ans.*	

SOLUTION.—In this example, 7, one part of the multiplier, is a factor of 35, the other part. We first find, in the usual manner, the product of the multiplicand by the 7 units; multiplying this product by 5, and writing the first figure of the result in tens' place, we obtain the product of the multiplicand by $7 \times 5 \times 10 = 35$ tens; and the sum of these two partial products, 1691823, must be the whole product required.

2. Multiply 58327 by 21318.

OPERATION.

58327	
21318	
174981	Prod. by 3 hundreds.
1049886	Prod. by 18 units.
1224867	Prod. by 21 thousands.
1243414986	*Ans.*

SOLUTION. — The 3 hundreds is a factor of 18, the part on the right of it, and also of 21, the part on the left of it. We first multiply by 3, writing the first figure in hundreds' place, because the 3 represents hundreds; multiplying this product by 6, and writing the first figure in units' place, we obtain the product of the multiplicand by $3 \times 6 =$ 18 units; multiplying the first partial product by 7, and writing the first figure in thousands' place, we obtain the product of the multiplicand by $7 \times 3 \times 1000 = 21$ thousands; the sum of these three partial products is the entire product.

3. Multiply 5643 by 4237.

OPERATION.

5643	
4237	
39501	
16929	
237006	
23909391	*Ans.*

SOLUTION. — The 7 units is a factor of 42. We first multiply by 7 units and by 3 tens, as in § 103. Then we multiply the first partial product by 6, and write the first figure in hundreds' place, since $4200 = 6 \times 7 \times 100$.

NOTE. — The product obtained by multiplying any partial product is called a *derived product.*

RULE. — I. *Find the product of the multiplicand by some term of the multiplier which is a factor of one or more parts of the multiplier.*

II. *Multiply this product by that factor which, taken with the term of the multiplier first used, will produce other parts of the multiplier, and write the first figure of each result under the first figure of the part of the multiplier thus used.*

III. *In like manner, find the product, either direct or derived, for every term or part of the multiplier; the sum of all the products will be the whole product required.*

Multiply:

4. 5784 by 246.
5. 3785 by 721.
6. 472856 by 54918.
7. 43785 by 9153.
8. 78563721 by 127369.
9. 43725652 by 5481918.
10. 3578426785 by 64532164.
11. 2703605 by 4249784.

133. When the multiplier is 9, 99, or any number of 9's.

Annexing one cipher to a number multiplies it by 10, two ciphers by 100, three ciphers by 1000, etc. Since 9 is $10 - 1$, any number may be multiplied by 9 by annexing 1 cipher to it and subtracting the number from the result (§ 123). For similar reasons, 100 times a number $-$ 1 time the number $=$ 99 times the number, etc.

1. Multiply 6556 by 999.

OPERATION.

$6556000 - 6556 = 6549444$ *Ans.*

SOLUTION. — Multiplying 6556 by 1000 (which is $999 + 1$) the result is 6556000. As this result is 1×6556 or 6556 too great, we subtract 6556 and the answer is 6549444.

RULE. — *Annex to the multiplicand as many ciphers as the multiplier contains 9's, and subtract the multiplicand from the result.*

2. $784 \times 99 = ?$
3. $587.3 \times 999 = ?$
4. $47.83 \times 99999 = ?$
5. $756 \times 999999 = ?$

134. When the multiplier is a number a few units less than the next higher unit.

If we wish to multiply by 97, which is $100 - 3$, we can evidently annex 2 ciphers to the multiplicand, and subtract 3 times the multiplicand from the result. If our multiplier is 991, which is $1000 - 9$, we can subtract 9 times the multiplicand from 1000 times the multiplicand.

1. Multiply 6556 by 993.

OPERATION.

$6556000 - (7 \times 6556) = 6556000 - 45892 = 6510108$ *Ans.*

SOLUTION. — We proceed as in Ex. 1, § 133, but since 1000 is 7 greater than 993, we must subtract 7×6556 from 6556000 and the answer is 6510108.

RULE. — *Multiply by the next higher unit by annexing ciphers. From this result subtract as many times the multiplicand as there are units in the difference between the multiplier and the next higher unit.*

2. $786 \times 93 = ?$
3. $4327 \times 96 = ?$
4. $7328 \times 997 = ?$
5. $7873.5 \times 995 = ?$
6. $43789 \times 9994 = ?$
7. $707.736 \times 999993 = ?$

135. When the left-hand figure of the multiplier is the unit 1, the right-hand figure is any digit whatever, and the intervening figures, if any, are ciphers.

1. Multiply 3684 by 17.

OPERATION.

$$\begin{array}{l} \underline{3684} \times 17 \\ 62628 \ \textit{Ans.} \end{array}$$

SOLUTION.—If we multiply by the usual method, we obtain, separately, 7 times and 10 times the multiplicand, and add them. We may therefore multiply by the 7 units, and to the product add the multiplicand regarded as tens, thus: 7 times 4 = 28, and we write the 8 as the unit figure of the product. Then, 7 times 8 = 56, and the 2 reserved being added = 58, and the 4 in the multiplicand, added, = 62, and we write 2 in the product. Next, 7 times 6, plus the 6 reserved, plus the 8 in the multiplicand, = 56, and we write 6 in the product. Next, 7 times 3, plus the 5 reserved, plus the 36 in the multiplicand, = 62, which we write in the product, and the work is done.

Had the multiplier been 107, we should have multiplied two figures of the multiplicand by 7, before we commenced adding the digits of the multiplicand to the partial products; 3 figures had the multiplier been 1007, etc.

RULE.—I. *Write the multiplier at the right of the multiplicand, with the sign of multiplication between them.*

II. *Multiply the multiplicand by the unit term of the multiplier, and to the product add the multiplicand, regarding its local value as a product by the left-hand term of the multiplier.*

Find the products:

2. 567 × 13.	5. 18075 × 1008.	8. 201.5 × 105.
3. 439603 × 105.	6. 390.7 × 10002.	9. 3.162 × 104.
4. 7859 × 107.	7. 143.01 × 10005.	10. 42.51 × 106.

136. When the left-hand figure of the multiplier is any digit, the right-hand figure is the unit 1, and the intermediate figures, if any, are ciphers.

1. Multiply 834267 by 301.

OPERATION.

$$\begin{array}{l} \underline{834267} \times 301 \\ 251114367 \ \textit{Ans.} \end{array}$$

SOLUTION.—Regarding the multiplicand as a product by the unit 1, of the multiplier, we multiply the multiplicand by 3 hundreds, and add the digits of the multiplicand to the several products as we proceed. Since the 3 is hundreds, the two right-hand figures of the mul-

tiplicand will be the two right hand figures of the product; and the product of 3×7 will be increased by 2, the hundreds of the multiplicand.

Had the multiplier been 31, the *tens* of the multiplicand would have been added to 3×7; had the multiplier been 3001 the *thousands* of the multiplicand would have been added to 3×7; and so on.

RULE.—I. *Write the multiplier at the right of the multiplicand, with the sign of multiplication between them.*

II. *Multiply the multiplicand by the left-hand term of the multiplier, and to the product add the multiplicand, regarding its local value as a product by the unit term of the multiplier.*

Multiply:

2. 56783 by 71.
3. 47.89 by 601.
4. 3724.5 by 901.
5. 103078 by 40001.

137. When the digits of the multiplier are all the same figure.

1. Multiply 81362 by 333.

OPERATION.

$$\begin{array}{r} 81362000 \\ 81362 \\ \hline 3)\overline{81280638} \\ \hline 27093546 \end{array} \; \textit{Ans.}$$

SOLUTION.—We first multiply by 999 (§ 133). Then, since 333 is $\frac{1}{3}$ of 999, we take $\frac{1}{3}$ of the product.

Had our multiplier been 444, we would have taken $\frac{4}{9}$ of 999 times the multiplicand. Had it been 66, we would have taken $\frac{6}{9} = \frac{2}{3}$ of 99 times the multiplicand.

RULE.—I. *Multiply by as many* 9's *as the multiplier contains digits, by* § 133.

II. *Take such a part of the product as* 1 *digit of the multiplier is part of* 9.

2. $432711 \times 222 = ?$
3. $578 \times 1111 = ?$
4. $.6732 \times 88888 = ?$
5. $867.5 \times 777 = ?$

138. To square a number consisting of only two digits.

1. What is the square of 18?

OPERATION.

$$18^2 = (16 \times 20) + 2^2$$
$$= 324 \; \textit{Ans.}$$

SOLUTION.—$18^2 = 18 \times 18$ (§ 108). Now if one of these factors is diminished by 2, the product will be less than the square of 18 by 2 times the other factor (§ 123, II); that is, $18^2 = (16 \times 18) + (2 \times 18)$. Next, if we increase the other factor, 18, in this result, by 2, the whole result will exceed the square of 18, by 2 times the other factor, 16 (§ 123, III); that is, $18^2 = (16 \times 20) + (2 \times 18) - (2 \times 16)$. But as 2 times 18 minus 2 times 16 is equal to 2×2, or 2^2, we have $18^2 = 16 \times 20 + 2^2$.

RULE. — I. *Take two numbers, one of which is as many units less than the number to be squared as the other is units greater, and one of the numbers taken an exact number of tens.*

II. *Find the product of these two numbers, and to it add the square of the difference between the given number and one of the assumed numbers.*

NOTE. — A little practice will enable the pupil to square mentally any number less than 100 by this rule.

Find the square of:

2. 27.	5. 26.	8. 37.	11. 77.	14. 69.	17. 59.
3. 49.	6. 39.	9. 36.	12. 88.	15. 68.	18. 83.
4. 28.	7. 38.	10. 35.	13. 99.	16. 62.	19. 97.

139. **To square any number consisting of 9's.**

1. Square 99, 999, 9999.

OPERATION.

$99 \times 99 = 9801$ *Ans.*
$999 \times 999 = 998001$ *Ans.*
$9999 \times 9999 = 99980001$ *Ans.*

SOLUTION. — If we perform the operation in the ordinary way, and compare the results, we find that in the first case we have for the digits *one* 9, an 8, *one* cipher, and 1; in the second, *two* 9's, an 8, *two* ciphers, and 1; in the third, *three* 9's, an 8, *three* ciphers, and 1. That is, each result consists of one 9 less than in the given number, an 8, the same number of ciphers as 9's, and a 1.

RULE. — *Write as many 9's less one as there are in the given number, an* 8, *as many ciphers as 9's, and a* 1.

2. Find the square of 99999.

3. What is the square of 999999?

4. What is 9 times 999999999^2?

140. An **Aliquot** or **Even Part** of a number is such a part as will exactly divide that number. Thus, 5, $8\frac{1}{3}$, and $12\frac{1}{2}$ are aliquot parts of 25 and of 100, etc.

NOTE. — An *aliquot part* may be either a whole or a mixed number, while a *component factor* must be a whole number.

The aliquot parts of 10 are 5, $3\frac{1}{3}$, $2\frac{1}{2}$, 2, $1\frac{2}{3}$, $1\frac{3}{7}$, $1\frac{1}{4}$, $1\frac{1}{9}$.

The aliquot parts of 100, 1000, or of any other number, may be found by dividing the number by 2, 3, 4, etc., until it has been divided by all the integral numbers between 1 and itself.

141. When the multiplier is an aliquot part of some higher unit.

1. Multiply 78 by $3\frac{1}{3}$, and by 25, separately.

OPERATION.

$$3\underline{)780} \qquad 4\underline{)7800}$$
$$260\ \textit{Ans.} \qquad 1950\ \textit{Ans.}$$

SOLUTION. — Since $3\frac{1}{3}$ is $\frac{1}{3}$ of 10, the next higher unit, we multiply 78 by 10 and take $\frac{1}{3}$ of the product.

Again, since 25 is $\frac{1}{4}$ of 100, we multiply 78 by 100 and take $\frac{1}{4}$ of the product.

RULE. — I. *Multiply the given multiplicand by the unit next higher than the multiplier by annexing ciphers.*

II. *Take such a part of this product as the given multiplier is part of the next higher unit.*

NOTE. — Since $6\frac{2}{3}$ is twice $3\frac{1}{3}$, we may multiply 78 by $6\frac{2}{3}$, by first multiplying by $3\frac{1}{3}$ by the rule just given, and then taking twice the result. In the same way we can multiply a number by any multiple of an aliquot part of a higher unit.

Multiply:

2. 437 by 25.
3. 6872 by $2\frac{1}{2}$.
4. 5734154 by $333\frac{1}{3}$.
5. 758642 by $12\frac{1}{2}$.
6. 78563 by 125.
7. 57687 by $142\frac{6}{7}$.

ALIQUOT PARTS OF 100.

$50 = \frac{1}{2}$ of 100.	$11\frac{1}{9} = \frac{1}{9}$ of 100.
$33\frac{1}{3} = \frac{1}{3}$ of 100.	$10 = \frac{1}{10}$ of 100.
$25 = \frac{1}{4}$ of 100.	$9\frac{1}{11} = \frac{1}{11}$ of 100.
$20 = \frac{1}{5}$ of 100.	$8\frac{1}{3} = \frac{1}{12}$ of 100.
$16\frac{2}{3} = \frac{1}{6}$ of 100.	$6\frac{1}{4} = \frac{1}{16}$ of 100.
$14\frac{2}{7} = \frac{1}{7}$ of 100.	$5 = \frac{1}{20}$ of 100.
$12\frac{1}{2} = \frac{1}{8}$ of 100.	$4 = \frac{1}{25}$ of 100.

MULTIPLES OF ALIQUOT PARTS OF 100.

$88\frac{8}{9} = \frac{8}{9}$ of 100.	$77\frac{7}{9} = \frac{7}{9}$ of 100.
$87\frac{1}{2} = \frac{7}{8}$ of 100.	$75 = \frac{3}{4}$ of 100.
$85\frac{5}{7} = \frac{6}{7}$ of 100.	$71\frac{3}{7} = \frac{5}{7}$ of 100.
$83\frac{1}{3} = \frac{5}{6}$ of 100.	$66\frac{2}{3} = \frac{2}{3}$ of 100.
$80 = \frac{4}{5}$ of 100.	$62\frac{1}{2} = \frac{5}{8}$ of 100.

$60 = \frac{3}{5}$ of 100.	$40 = \frac{2}{5}$ of 100.
$57\frac{1}{7} = \frac{4}{7}$ of 100.	$37\frac{1}{2} = \frac{3}{8}$ of 100.
$55\frac{5}{9} = \frac{5}{9}$ of 100.	$28\frac{4}{7} = \frac{2}{7}$ of 100
$44\frac{4}{9} = \frac{4}{9}$ of 100.	$22\frac{2}{9} = \frac{2}{9}$ of 100.
$42\frac{6}{7} = \frac{3}{7}$ of 100.	$18\frac{2}{11} = \frac{2}{11}$ of 100.

Aliquot Parts of 1000.

$500 = \frac{1}{2}$ of 1000.	$111\frac{1}{9} = \frac{1}{9}$ of 1000.
$333\frac{1}{3} = \frac{1}{3}$ of 1000.	$100 = \frac{1}{10}$ of 1000.
$250 = \frac{1}{4}$ of 1000.	$90\frac{10}{11} = \frac{1}{11}$ of 1000.
$200 = \frac{1}{5}$ of 1000.	$83\frac{1}{3} = \frac{1}{12}$ of 1000.
$166\frac{2}{3} = \frac{1}{6}$ of 1000.	$62\frac{1}{2} = \frac{1}{16}$ of 1000.
$142\frac{6}{7} = \frac{1}{7}$ of 1000.	$50 = \frac{1}{20}$ of 1000.
$125 = \frac{1}{8}$ of 1000.	$40 = \frac{1}{25}$ of 1000.

142. When the right-hand term or terms of the multiplier are aliquot parts of 10, 100, 1000, etc.

1. Multiply 2183 by $1233\frac{1}{3}$.

OPERATION.

$$\begin{array}{l} 218300 \\ \underline{\quad 12\frac{1}{3}} \\ \quad 72766\frac{2}{3} \\ \underline{26196} \\ 2692366\frac{2}{3} \ \textit{Ans.} \end{array}$$

Solution. — Since $1233\frac{1}{3} = 12\frac{1}{3} \times 100$, we multiply by 100, and by $12\frac{1}{3}$, in continued multiplication (§ 124).

Rule. — I. *Reject from the right hand of the multiplier such term or terms as are an aliquot part of some higher unit, and to the remaining terms of the multiplier annex a fraction which expresses the aliquot part thus rejected, for a reserved multiplier.*

II. *Annex to the multiplicand as many ciphers as are equal to the number of terms rejected from the right hand of the multiplier, and multiply the result by the reserved multiplier.*

Multiply:

2. 43789 by 825.
3. 58730 by 7125.
4. 7854 by $362\frac{1}{2}$.
5. 30724 by $73333\frac{1}{3}$.
6. 47836 by $712\frac{1}{2}$.
7. 53727 by $2416\frac{2}{3}$.

143. To find the cost of a quantity when the price is an aliquot part of a dollar.

1. What is the cost of a case of muslins containing 1624 yd., at $\$.12\frac{1}{2}$ per yard?

OPERATION.	
$8\underline{)\$1624}$	SOLUTION. — At $1 per yard the case would cost $1624. As $\$.12\frac{1}{2} = \$\frac{1}{8}$, it will cost $\frac{1}{8}$ of $1624, or $203.
$203 *Ans.*	

RULE. — *Take such a part of the given quantity as the price is part of one dollar.*

2. What is the cost of 568 pounds of butter at 25 cents a pound?

3. What is the cost of 28 dozen candles at $\$.12\frac{1}{2}$ a dozen?

4. What is the cost of 576 pounds of beef at $11\frac{1}{9}$ cents a pound?

5. What is the cost of 64 pieces of ribbon, each containing 10 yards at $\$.12\frac{1}{2}$ a yard?

6. What is the cost of 12 dozen buttons at $\$.37\frac{1}{2}$ a dozen?

ACCOUNTS AND BILLS.

144. An **Account** is a registry of debits and credits.

NOTES. — 1. An account should always contain the names of both the parties to the transaction, the price or value of each item or article, and the date of the transaction.

2. Accounts may have only one side, which may be either debit or credit; or it may have two sides, debit and credit. (For the balance of an account, see § 90.)

145. An **Account Current** is a full copy of an account, giving each item of both debit and credit sides to date.

146. A **Bill**, in business transactions, is a written statement of articles bought or sold, with the prices of each and the total cost; or a statement of services rendered, with the price or value annexed to each item.

147. The **Footing of a Bill** is the total amount or cost of all the items.

NOTE. — A bill of goods bought or sold, or of services received or rendered at a single transaction, and containing only one date, is often called a *bill of parcels;* and an account current having only one side is sometimes called a *bill of items.*

148. In accounts and bills the following abbreviations are in general use:

@, at.	*Cr.*, creditor.	*M.*, one thousand.
%, *acc't*, account.	*cwt.*, hundredweight.	*mo.*, months.
bal., balance.	*Dr.*, debtor.	*pay't*, payment.
bbl., barrels.	*doz.*, dozen.	*pr.*, pair.
bu., bushels.	*hhd.*, hogshead.	*rec'd*, received.
C., one hundred.	*lb.*, pound.	*yd.*, yards.

149. When an account current or a bill is settled or paid, the fact should be entered on the same and signed by the creditor, or by the person acting for him. The % or bill is then said to be *receipted.* Accounts and bills may be settled and receipted by the parties to the same, or by agents, clerks, or attorneys authorized to transact business for them.

Examples.

150. Find the cost of the several articles, and the amount or footing of the following bills:

1.

Bill, receipted by clerk or agent.

New York, July 10, 1895.

Mr. John C. Hale,

Bought of Hill, Groves & Co.,

10 yd.	Cassimere,	@	$2.85	
16 "	Blk. Silk,	"	1.12½	
72 "	Ticking,	"	.14	
42 "	Shirting,	"	.16½	
12 "	Flannel,	"	.40	
24½ "	India Silk,	"	.56	
14 "	Alpaca,	"	.55	
5 "	Lace,	"	1.25	

Rec'd Payment,

Hill, Groves & Co.,

By J. W. Hopkins.

2.

Bill, receipted by the selling party.

CHICAGO, Sept. 20, 1894.

iase & Kennard,

Bought of McDougal, Fenton & Co.,

125	pr.	Boys' Thick Boots,	@	$1.25
275	"	" Calf "	"	4.25
180	"	" Kip "	"	3.15
80	"	Women's Fox'd Gaiters,	"	3.50
95	"	" Opera Boots,	"	2.75
8	cases	Men's Calf Boots,	"	30.50
3	"	Congress Pump Boots,	"	35.75
40	gross	Silk Buttons,	"	.37½

Rec'd Payment,

McDougal, Fenton & Co.

3.

Bill, settled by note.

NEW YORK, May 4, 1894.

ith & Perkins,

Bought of Kent, Lowber & Co.,

40	chests	Green Tea,	@	$27.50
25	"	Black "	"	19.20
16	"	Imperial "	"	48.10
12	sacks	Java Coffee,	"	17.75
20	bbl.	Coffee Sugar (A),	"	26.30
15	"	Crushed "	"	31.85
36	boxes	Lemons,	"	3.87½
42	"	Oranges,	"	4.12½
25	"	Raisins,	"	2.90
32	bbl.	Apples,	"	2.50

Rec'd Payment, by note at 6 mo.,

Kent, Lowber & Co.

4.

Bill, paid by draft, and receipted by clerk.

New Orleans, April 28, 1895.

James Carlton & Co.,

Bought of Willard & Hale,

150 bbl.	Canada Flour,	@	$ 5.50		
275 "	Genesee "	"	6.00		
170 "	Philada. "	"	5.00		
326 bu.	Wheat,	"	.85		
214 "	Corn,	"	.55		
300 "	Oats,	"	.35		

Rec'd Payment, by draft on N. Y.

R. S. Clarke,
For Willard & Hale.

5.

Account Current, not balanced or settled.

Philadelphia, Nov. 1, 1894.

Mr. James Cornwall,

To Dodge & Son, *Dr.*

April 15,	To	24 Tons Swedes Iron,	@	$ 64.30		
" "	"	15 cwt. Eng. Blister Steel,	"	10.25		
June 21,	"	7 doz. Hoes (Trowel Steel),	"	7.78		
Aug. 10,	"	25 " Buckeye Plows	"	10.75		
Oct. 3,	"	14 Cross-cut Saws,	"	16.12½		
" 3,	"	27 cwt. Bar Lead,	"	5.90		
" 3,	"	1840 lb. Chain,	"	.09½		

Cr.

May 25,	By	20 M. Boards,	@	$ 17.60	
July 14,	"	50 M. Shingles,	"	3.12½	
" 14,	"	42 M. Plank,	"	9.37½	
Sept. 5,	"	Draft on New York,			$ 1000
" 12,	"	75 C. Timber,	@	3.10	
" 12,	"	36 C. Scantling,	"	.87½	

Balance Due Dodge & Son,

6.

Account Current, another form, balanced by note.

WM. RICHMOND & CO. IN % CURRENT WITH WOOD & POWELL.

Dr. *Cr.*

1895					1895				
July	2	To 596 pounds butter, @$.28			Nov	8	By 61 barrels apples, $2.25		
Aug.	17	" 373 " cheese, " .10			"	24	" 70 bushels turnips, .22		
"	24	" 481 " lard, " .12½			Dec	1	" 56 " dried apples, .37½		
Oct.	4	" 509 " tallow, " .15			"	22	" 31 drums figs, .08		
"	18	" 81 dozen eggs, " .16⅔			1896				
"	31	" 15 barrels salt, " 1.40			Jan	2	" Note at 3 mo. to Bal.		
Dec.	15	" 41 hams, 963 lb., " .12½							

BOSTON, JAN. 1, 1896. WOOD & POWELL.

7.

NEW YORK, June 1, 1895.

MRS. T. HOWARD,

To B. ALTMAN & CO.,

DRY GOODS.

May	*6*	*3 Gloves*	*$1.35*		
		1 Box Paper	*.48*		
	7	*6 Ribbon*	*.05*		
		4 "	*.16*		
		7 Crepon	*.95*		
		2¾ Damask $1.25 for	*2.80*		
	9	*1 Velveteen Binding*	*.16*		
		1 Doz. Bones	*1.36*		
	10	*1 " Buttons*	*.12*		
	14	*3 Shirts*	*1.25*		
	23	*1 Gloves*	*.95*		
		4 Skeins Silk	*.03*		
		3½ Mohair	*.50*		
		Credit			
	9	*1 Gloves*	*1.35*		
	16	*3 Shirts*	*1.25*		
	25	*3½ Mohair*	*.50*		

Rec'd payment June 5, 1895

B. ALTMAN & CO.
Per

PROOFS OF MULTIPLICATION.

151. To prove multiplication by varying the partial products.

This proof is based on the principle that the product of any two factors is the same, whichever is taken as a multiplier (§ 102, Prin. I).

RULE.—*Invert the order of the factors, that is, multiply the multiplier by the multiplicand. If the product is the same as the first answer, the work is correct.*

152. To prove multiplication by casting out the 9's.

Since every number is equal to an exact number of 9's plus the sum of its digits § 77, if the excess of 9's in the digits is subtracted from a number the remainder must be an exact number of 9's. Hence we have the following principles:

PRINCIPLES.—I. *If the excess of 9's is subtracted from a number, the remainder will be a number having no excess of 9's.*

II. *If a number having no excess of 9's is multiplied by any number, the product will have no excess of 9's.*

1. Prove by excess of 9's that 473 multiplied by 138 = 65274.

EXPLANATION.

$$473 = 468 + 5$$
$$138 = 135 + 3$$

$$\text{Partial products, }\left\{\begin{array}{rcr} 468 \times 135 &=& 63180 \\ 5 \times 135 &=& 675 \\ 468 \times 3 &=& 1404 \\ 5 \times 3 &=& 15 \end{array}\right.$$

$$\text{Entire product, } \quad 65274$$

OPERATION.

473,	exc. 5		
138,	exc. 3		
65274,	exc. 6	15,	exc. 6.

SOLUTION.—The excess of 9's in 473 is 5, and 473 = 468 + 5, of which the first part, 468, contains no excess of 9's (I). The excess of 9's in 138 is 3, and 138 = 135 + 3, of which the first part, 135, contains no excess of 9's (I). Multiplying both parts of the multiplicand by each part of the multiplier, we have four partial products, the first three have no excess of 9's, because each contains a factor having no excess of 9's (II). Therefore, the excess of 9's in the entire product must be the same as the excess of 9's in the last partial product, 15, which we find to be 6. The same may be shown of any two numbers.

RULE.—*From the product of the excess of 9's in the multiplicand and multiplier, cast out the 9's. Compare the excess with the excess in the answer.*

153. To prove multiplication by casting out the 11's.

1. Prove by excess of 11's that 473 multiplied by 138 = 65274.

OPERATION.

473,	$7 - 7 = 0$
138,	$9 - 3 = 6$
65274	$\underline{0}$
$12 - 12 = 0$, exc.	exc. $= 0$

SOLUTION. — This proof depends upon the same principles as govern the proof by excess of 9's. We find by § 78 that the excess of 11's in the multiplicand is 0, in the multiplier 6, their product is $0 \times 6 = 0$. The excess of 11's in the product 65274 is also 0, hence that answer is probably correct.

RULE. — *From the product of the excess of* 11's *in the multiplicand and multiplier cast out the* 11's. *Compare the excess with the excess in the answer.*

NOTE. — If the excess of 9's or 11's in either factor is 0, the excess of 9's or 11's in the product will be 0 (§ 152, II).

154. To prove multiplication by summing up the digits.

1. Prove by summing up the digits that 473 multiplied by 138 = 65274.

OPERATION.

473,	5
138,	3
65274, 6 final dig.	15, 6 final dig.

SOLUTION. — Summing up the digits in the multiplicand (§ 79) the product is 5; in the multiplier 3. Their product, 15, summed up = 6, the final digit. The final digit in 65274 = 6. Hence the answer is probably correct.

RULE. — *Sum up the multiplicand and the multiplier and find the final digit of the product. Compare this with the final digit of the answer.*

Examples combining the Preceding Rules.

155. 1. A man bought two farms, one containing 175 acres, at $ 28 per acre, and the other containing 320 acres, at $ 37 per acre. What was the cost of both?

2. If a man receives $ 1200 yearly salary, and pays $ 364 for board, $ 275 for clothing, $ 150 for books, and $187 for other expenses, how much can he save in 5 years?

3. Two persons start from the same point, and travel in opposite directions; one travels 29 miles a day, and the other 32 miles. How far apart will they be in 17 days?

4. A drover who bought 127 head of cattle, at \$ 34 a head, and 97 head, at \$ 47 a head, sold the whole lot at \$ 40 a head. What was his entire profit or loss?

5. The lowest standard amount of fresh air needed for each person is 45 cubic feet a minute. If a schoolroom contains 50 pupils, how much fresh air must pass through it during a session of 3 hours to keep the atmosphere pure?

6. Find the value of $2^4 \times 5^5 - 7^3$.

7. $15^3 - (3^2 \times 2^5) + 208^2 - 9 \times 2^4$.

8. If a house is worth \$ 2450, and the farm on which it stands 6 times as much, lacking \$ 500, and the stock on the farm twice as much as the house, what is the value of the whole?

9. A man invests in trade at one time \$ 450, at another \$ 780, at another \$ 1250, and at another \$ 2275. How much must he add to these sums, that the amount invested by him originally shall be increased fourfold?

10. In a certain city there are 997 public schools, each containing an average of 665 pupils. How many public school pupils are there in that city?

11. If a steam car travels at the rate of 625 miles a day, how many miles will it cover in 19 days? in 109 days?

12. The cost of the Atlantic Telegraph Cable, as originally made, was as follows: 2500 miles at \$ 485 per mile, 10 miles deep-sea cable at \$ 1450 per mile, and 25 miles shore ends at \$ 1250 per mile. What was its total cost?

13. What is the area of a square lot which is 59 ft. long?

14. A merchant sold 54 yards of calico at \$ $.16\frac{2}{3}$ per yard, 16 pieces of sheeting, each piece containing 33 yards, at \$ $.06\frac{1}{4}$ per yard, and received in payment 18 bushels of oats, at \$ $.33\frac{1}{3}$ per bushel, and the balance in money. How much money did he receive?

DIVISION.

156. **Division** is the process of separating a number into equal parts, or of finding how many times one number is contained in another.

157. The **Dividend** is the number to be divided.

158. The **Divisor** is the number to divide by.

159. The **Quotient** is the result obtained by the division.

160. The **Reciprocal** of a number is 1 divided by the number. Thus, the reciprocal of 15 is $1 \div 15$, or $\frac{1}{15}$.

NOTES. — 1. When the dividend does not contain the divisor an exact number of times, the part of the dividend left is called the *Remainder*, and this must be less than the divisor.

2. As the remainder is always a part of the dividend, it is always of the same name or kind. When there is no remainder the division is said to be *exact*.

161. In separating a number into equal parts, the *divisor is always abstract*, and the dividend and quotient are like numbers. This is sometimes called the **Partitive** form of division.

Thus, if we wish to divide 35 cents equally among 7 children, each child will receive one of 7 equal parts into which 35 cents may be divided, or $\frac{1}{7}$ of 35 cents, which is 5 cents. 35 cents $\div$ 7 = 5 cents. The dividend and quotient are like numbers, and the divisor is abstract.

162. In finding how many times one number is contained in another, the divisor and dividend are like numbers (concrete or abstract), and the *quotient is always abstract.* This is sometimes called the **Measuring** form of division.

Thus, if we wish to find how many oranges can be bought for 35 cents, if each costs 7 cents, we say as many oranges can be bought as 7 cents is contained times in 35 cents, which is 7 times. 35¢ $\div$ 5¢ = 7. The divisor and dividend are like numbers (*cents*), and the quotient is abstract.

163. The method of dividing any number by another depends upon the following principles:

PRINCIPLES. — I. *Division is the reverse of multiplication, the dividend corresponding to the product, and the divisor and quotient to the factors.*

II. *If all the parts of a number are divided, the entire number will be divided.*

III. *Since the remainder in dividing any part of the dividend must be less than the divisor, it can be divided only by being expressed in units of a lower order; hence the operation must commence with the units of the highest order.*

Examples.

164. 1. Divide 2742 by 6.

OPERATION.

$$6\overline{)2742}$$
$$457 \; Ans.$$

SOLUTION. — We write the divisor at the left of the dividend, separated from it by a line. As 6 is not contained in 2 thousands, we take the 2 thousands and 7 hundreds together, and proceed thus: 6 is contained in 27 hundreds 4 hundreds times, and the remainder is 3 hundreds; we write 4 in hundreds' place in the quotient, and unite the remainder, 3 hundreds, to the next figure of the dividend, making 34 tens; then, 6 is contained in 34 tens 5 tens times, and the remainder is 4 tens; writing 5 tens in its place in the quotient, we unite the remainder to the next figure in the dividend, making 42; 6 is contained in 42 units 7 times, and there is no remainder; writing 7 in its place in the quotient, we have the entire quotient, 457.

NOTE. — The different numbers which we divide in obtaining the successive figures of the quotient, are called *partial dividends*.

2. Divide 18149 by 56.

OPERATION.

```
56) 18149 (324 5/56 Ans.
    168
    ---
     134
     112
     ---
      229
      224
      ---
        5
```

SOLUTION. — As neither 1 nor 18 will contain the divisor, we take three terms, 181, for the first partial dividend. 56 is contained in 181, 3 times, with a remainder; we write the 3 as the first figure in the quotient, and then multiply the divisor by this quotient term; 3 times 56 = 168, which, subtracted from 181, leaves 13; to this remainder we annex or bring down 4, the

next term of the dividend, and thus form 134, the next partial dividend; 56 is contained in 134, 2 times, with a remainder; 2 times 56 = 112, which, subtracted from 134, leaves 22; to this remainder we bring down 9, the last term of the dividend, and we have 229, the last partial dividend; 56 is contained in 229, 4 times, with a remainder; 4 times 56 = 224, which, subtracted from 229, gives 5, the final remainder, which we write in the quotient with the divisor below it, thus completing the division.

NOTE. — When the multiplication and subtraction are performed mentally, as in the first example, the operation is called *Short Division;* but when the work is written out in full, as in the second example, the operation is called *Long Division.* The principles governing the two methods are the same.

3. Divide 181.44 by 56.

OPERATION.

```
56) 181.44 (3.24 Ans.
    168
    ---
     134
     112
     ---
      224
      224
      ---
```

SOLUTION. — 56 is contained in 181, 3 times, with a remainder of 13 units. To this remainder we annex the next term of the dividend, which is 4 tenths, thus making 134 tenths; 56 is contained in 134 tenths, 2 tenths times, with a remainder; so we place 2 in the quotient, preceded by the decimal point. To the remainder, 22 tenths, we add the 4 hundredths of the dividend, making 224 hundredths; 56 is contained in 224 hundredths, 4 hundredths times, without a remainder, so we place 4 in hundredths' place in the quotient, and the answer is 3.24.

RULE. — I. *Beginning at the left hand, take for the first partial dividend the fewest terms of the given dividend that will contain the divisor one or more times; find how many times the divisor is contained in this partial dividend, and write the result in the quotient; multiply the divisor by this quotient term, and subtract the product from the partial dividend used.*

II. *To the remainder bring down the next term of the dividend, with which proceed as before; and thus continue till all the terms of the dividend have been divided.*

III. *In dividing a decimal by a whole number, place a decimal point in the quotient as soon as the decimal point in the dividend is reached, or point off as many decimal places in the quotient as there are in the dividend.*

IV. *If the division is not exact, place the final remainder in the quotient, and write the divisor underneath.*

NOTE. — For the rule for dividing a decimal by a decimal see § 290.

Divide:

4. 6) 473832.
5. 8) 972496.
6. 9) 1370961.
7. 12) 73042164.
8. 170352 by 36.
9. 40988.7 by 47.
10. 443520 by 84.
11. 36381.25 by 125.
12. 1554768 by 216.
13. 4828.8058 by 3094.
14. 112.1488 by 232.
15. 27085946 by 216.
16. 29137.16 by 5317.
17. 49179.2525 by 2359.

What is the value of:

18. 3931.476 ÷ 556?
19. 721198 ÷ 291?
20. 3844449 ÷ 657?
21. 536819.1968 ÷ 907?
22. 571938.5742 ÷ 37149?
23. 48659910 ÷ 54001?

24. The annual receipts of a manufacturing company are $ 147675. How much is that per day, there being 365 days in the year?

25. A man who fails in business owes $ 115275. His property is valued at $ 40346.25. How much can he pay on a dollar?

26. There are 5280 ft. in a mile. How many miles are equivalent to 295680 ft.?

27. The telegraph poles along a railroad are set 264 ft. apart. In passing by train I counted 82 poles in 10 minutes. How many miles an hour was the train running?

28. If a bank pays me $ 16875 interest on $ 281250 for 1 year, how much will it pay on $ 6250 for the same time?

29. If a tax of $ 72320060.75 is equally levied on 10735 towns, what amount must each town pay?

30. If there are 699425 books in a number of libraries, and each has an average of 27977 books, what is the number of libraries?

31. The world contains an area of 52361115 square miles, and its population is estimated at 1479486192. How many people are there in the world to a square mile?

GENERAL PRINCIPLES OF DIVISION.

165. The general principles of division most important in their application, relate: first, to changing the terms of division by addition or subtraction; second, to changing the terms of division by multiplication or division; third, to successive division.

The quotient in division depends upon the relative values of the dividend and divisor. Hence, any change in the value of either dividend or divisor must produce a change in the value of the quotient; though certain changes may be made in both dividend and divisor at the same time, that will not affect the quotient.

166. Changing the terms by addition or subtraction.

Since the dividend corresponds to a product, of which the divisor and quotient are factors, we observe,

1. If the divisor is *increased* by 1, the dividend must be increased by as many units as there are in the quotient, in order that the quotient may remain the same (§ 123, I); and if the dividend is *not* thus increased, the quotient will be *diminished* by as many units as the number of times the new divisor is contained in the quotient.

$$84 \div 6 = 14. \qquad 84 \div 7 = 14 - \tfrac{14}{7} = 12.$$

2. If the divisor is *diminished* by 1, the dividend must be diminished by as many units as there are in the quotient, in order that the quotient may remain the same (§ 123, II); and if the dividend is *not* thus diminished, the quotient will be *increased* by as many units as the number of times the new divisor is contained in the quotient.

$$144 \div 9 = 16. \qquad 144 \div 8 = 16 + \tfrac{16}{8} = 18.$$

167. These principles may be stated as follows:

PRINCIPLES.—I. *Adding* 1 *to the divisor takes as many units from the quotient as the new divisor is contained times in the quotient.*

II. *Subtracting* 1 *from the divisor adds as many units to the quotient as the new divisor is contained times in the quotient.*

III. ADDING *any number to the divisor* SUBTRACTS *as many units from the quotient as the new divisor is contained times in the product of the quotient by the number added; and* SUBTRACTING *any number from the divisor* ADDS *as many units to the quotient as the new divisor is contained times in the product of the quotient by the number subtracted.*

168. Changing the terms by multiplication or division.

1. If any divisor is contained in a given dividend a certain number of times, the same divisor will be contained in twice the dividend twice as many times; in three times the dividend, three times as many times; and so on.

I. *Multiplying the dividend by any number multiplies the quotient by the same number.*

2. If any divisor is contained in a given dividend a certain number of times, the same divisor will be contained in one half the dividend one half as many times; in one third the dividend, one third as many times; and so on.

II. *Dividing the dividend by any number divides the quotient by the same number.*

3. If a given divisor is contained in any dividend a certain number of times, twice the divisor will be contained in the same dividend one half as many times; three times the divisor, one third as many times; and so on.

III. *Multiplying the divisor by any number divides the quotient by the same number.*

4. If a given divisor is contained in any dividend a certain number of times, one half the divisor will be contained in the same dividend twice as many times; one third of the divisor, three times as many times; and so on.

IV. *Dividing the divisor by any number multiplies the quotient by the same number.*

5. If a given divisor is contained in a given dividend a certain number of times, twice the divisor will be contained the same number of times in twice the dividend; three times the divisor will be contained the same number of times in three times the dividend; and so on.

V. *Multiplying both dividend and divisor by the same number does not alter the quotient.*

6. If a given divisor is contained in a given dividend a certain number of times, one half the divisor will be contained the same number of times in one half the dividend; one third of the divisor will be contained the same number of times in one third of the dividend; and so on.

VI. *Dividing both dividend and divisor by the same number does not alter the quotient.*

NOTE.—If a number is multiplied and the product divided by the same number, the quotient will be equal to the number multiplied; hence the fifth case may be regarded as a direct consequence of the first and third; and the sixth, as the direct consequence of the second and fourth.

To illustrate these cases, take 24 for a dividend and 6 for a divisor; then the quotient will be 4, and the several changes may be represented in their order as follows:

	Dividend.		Divisor.		Quotient.	
	24	÷	6	=	4	
1.	48	÷	6	=	8	Multiplying the dividend by 2 multiplies the quotient by 2.
2.	12	÷	6	=	2	Dividing the dividend by 2 divides the quotient by 2.
3.	24	÷	12	=	2	Multiplying the divisor by 2 divides the quotient by 2.
4.	24	÷	3	=	8	Dividing the divisor by 2 multiplies the quotient by 2.
5.	48	÷	12	=	4	Multiplying both dividend and divisor by 2 does not alter the quotient.
6.	12	÷	3	=	4	Dividing both dividend and divisor by 2 does not alter the quotient.

169. These six cases constitute three general principles, which may now be stated as follows:

PRINCIPLES.—I. *Multiplying the dividend multiplies the quotient, and dividing the dividend divides the quotient.*

II. *Multiplying the divisor divides the quotient, and dividing the divisor multiplies the quotient.*

III. *Multiplying or dividing both dividend and divisor by the same number does not alter the quotient.*

170. These three principles may be embraced in one general law:

GENERAL LAW.—*A change in the* DIVIDEND *produces a* LIKE *change in the quotient; but a change in the* DIVISOR *produces an* OPPOSITE *change in the quotient.*

171. Successive Division is the process of dividing one number by another, and the resulting quotient by a second divisor, and so on. It is the reverse of continued multiplication.

172. Successive division is based on the following principles:

PRINCIPLES. — I. *If a given number is divided by several numbers in successive division, the result will be the same as if the given number were divided by the product of the several divisors* (§ 125, I).

II. *The result of successive division is the same in whatever order the divisors are taken* (§ 125, II).

CONTRACTIONS IN DIVISION.

Examples.

173. To abbreviate long division.

We may avoid writing the products in long division, and obtain the successive remainders by the method of subtraction employed in the case of several subtrahends (§ 87).

1. Divide 261249 by 487.

OPERATION.

```
            Quo.
487)261249(536
      177
       313
        217 Rem.
```

SOLUTION. — Dividing the first partial dividend, 2612, we obtain 5 for the first term of the quotient. We then multiply 487 by 5; but instead of writing the product, and subtracting it from the partial dividend, we simply observe what numbers must be added to the product, as we proceed, to give the terms of the partial dividend, and write them for the remainder sought. Thus, 5 times 7 are 35, and 7 (written in the remainder) are 42, a number whose unit term is the same as the right-hand term of the partial dividend; 5 times 8 are 40, and 4, the tens of the 42, are 44, and 7 (written in the remainder) are 51; 5 times 4 are 20, and 5, the tens of the 51, are 25, and 1 (written in the remainder) are 26. We next consider the whole remainder, 177, as joined with 4, the next term of the dividend, making 1774 for the next partial dividend. Proceeding as before, we obtain 313 for the second remainder, 217 for the final remainder, and 536 for the entire quotient.

RULE. — I. *Obtain the highest term in the quotient in the usual manner.*

II. *Multiply the unit term of the divisor by this quotient term, and write such a number in the remainder as, added to this*

partial product, will give an amount having for its unit term the first or right-hand term of the partial dividend used.

III. *Carry the tens' term of the amount to the product of the next term of the divisor, and proceed as before, till the entire remainder is obtained.*

IV. *Conceive this remainder to be joined to the next term of the dividend for a new partial dividend, and proceed as with the former till the work is finished.*

Divide:

2. 77112 by 204.
3. 65664 by 72.
4. 7913576 by 209.
5. 6636584 by 698.
6. 4024156 by 8903.
7. 760592 by 6791.
8. 101443929 by 25203.
9. 1246038849 by 269181.
10. 2318922 by 56240.
11. 1454900 by 17300.

174. When the divisor is a composite number.

1. Divide 1242 by 54.

OPERATION.

$$\begin{array}{r} 6\underline{)1242} \\ 9\underline{)207} \\ 23 \end{array} \; \textit{Ans.}$$

SOLUTION. — The component factors of 54 are 6 and 9. We divide 1242 by 6, and the resulting quotient by 9, and obtain for the final result, 23, which must be the same as the quotient of 1242 divided by 6 times 9, or 54 (§ 172, I). We might have obtained the same result by dividing first by 9, and then by 6 (§ 172, II).

RULE. — *Divide the dividend by one of the factors, and the quotient thus obtained by another, and so on if there are more than two factors, until every factor has been made a divisor. The last quotient will be the quotient required.*

What is the value of:

2. $7315 \div 35$.
3. $48384 \div 54$.
4. $38850 \div 75$.
5. $62937 \div 189$.
6. $182250 \div 120$.
7. $181440 \div 504$.
8. $101088 \div 972$.
9. $850500 \div 180$.
10. $155925 \div 2079$.
11. $466560 \div 1296$.

175. To find the true remainder.

If remainders occur in successive division, it is evident that the true remainder must be the least number which, subtracted from the given dividend, will render all the divisions exact.

1. Divide 5855 by 168, using the factors 3, 7, and 8, and find the true remainder.

OPERATION.

3)5855	
7)1951	2
8)278 . . . . $5 \times 3 =$	15
34 . . $6 \times 7 \times 3 =$	126
True remainder	143

SOLUTION. — Dividing the given dividend by 3, we have 1951 for a quotient, and a remainder of 2. Hence, 2 subtracted from 5855 would render the first division exact, and we therefore write 2 for a part of the true remainder. Dividing 1951 by 7, we have 278 for a quotient, and a remainder of 5. Hence, 5 subtracted from 1951 would render the second division exact. But to diminish 1951 by 5 would require us to diminish 1951×3, the dividend of the first exact division, by $5 \times 3 = 15$ (§ 123, III); and we therefore write 15 for the second part of the true remainder. Dividing 278 by 8, we have 34 for a quotient, and a remainder of 6. Hence, 6 subtracted from 278 would render the third division exact. But to diminish 278 by 6 would require us to diminish 278×7, the dividend of the second exact division, by 6×7; or $278 \times 7 \times 3$, the dividend of the first exact division, by $6 \times 7 \times 3 = 126$; and we therefore write 126 for the third part of the true remainder. Adding the three parts, we have 143 for the entire remainder.

RULE. — I. *Multiply each partial remainder by all the preceding divisors.*

II. *Add the several products; the sum will be the true remainder.*

Perform the division by factoring, and find the true remainder.

2. $439 \div 15$.
3. $4270 \div 56$.
4. $17856 \div 72$.
5. $15288 \div 42$.
6. $972562 \div 168$.
7. $526050 \div 126$.
8. $612360 \div 105$.
9. $553 \div 15$.
10. $10183 \div 105$.
11. $10199 \div 120$.
12. $29795 \div 144$.
13. $73522 \div 168$.
14. $63844 \div 135$.
15. $386639 \div 720$.

176. When the divisor is 10, 100, 1000, 10000, etc.

If we cut off or remove the right-hand figure of a number, each of the other figures is removed one place toward the right, and, consequently, the value of each term is diminished tenfold, or divided by 10 (§ 52). For a similar reason, by cutting off *two* figures we divide by 100; by cutting off *three*, by 1000, etc.; and the figures cut off will constitute the remainder.

1. Divide 26825 by 1000.

OPERATION.

$26825 \div 1000 =$
$26|825 = 26\frac{825}{1000}$.

SOLUTION. — Since there are three ciphers in the divisor, we cut off three figures from the dividend, which constitute the remainder, and the answer is $26\frac{825}{1000}$.

RULE. — *From the right hand of the dividend cut off as many figures as there are ciphers in the divisor. Under the figures so cut off, place the divisor, and the whole will represent the quotient.*

Divide:

2. 79 by 10.
3. 7982 by 100.
4. 4003 by 1000.
5. 2301.05 by 10000.
6. 36000.36 by 1000.
7. 2373.509 by 100.

177. When there are ciphers on the right hand of the divisor.

1. Divide 25548 by 700.

OPERATION.

$7|00)\underline{255|48}$
36, Quotient. 3, 2d rem.
$3 \times 100 + 48 = 348$, true rem.

SOLUTION. — We resolve 700 into the factors 100 and 7. Dividing first by 100, the quotient is 255, and the remainder 48. Dividing 255 by 7, the final quotient is 36, and the second remainder 3. Multiplying the last remainder, 3, by the preceding divisor, 100, and adding the preceding remainder, we have $300 + 48 = 348$, the true remainder (§ 175). In practice, the true remainder may be obtained by prefixing the second remainder to the first.

RULE. — I. *Cut off the ciphers from the right of the divisor, and the same number of figures from the right of the dividend.*

II. *Divide the remaining terms of the dividend by the remaining terms of the divisor, for the final quotient.*

III. *Prefix the remainder to the figures cut off, and the result will represent the true remainder.*

Divide:

2. 7856 by 900.
3. 13872 by 500.
4. 83248 by 2600.
5. 1548036 by 4300.
6. 436000 by 300.
7. 66472000 by 8100.
8. 10818000 by 3600.
9. 9756.84 by 800.

178. When the figure on the right hand of the divisor is 5.

1. Divide 6545 by 55.

OPERATION.

$$\begin{array}{r|l} 55 & 6545 \\ 2 & 2 \\ \hline 11|0 & 1309|0 \\ & 119 \ \textit{Ans.} \end{array}$$

SOLUTION. — If we multiply both dividend and divisor by 2, the quotient will not be changed (§ 169, III), and we shall have a cipher on the right hand of the divisor. We then proceed to divide as in § 177.

RULE. — *Multiply both dividend and divisor by 2 and proceed as in* § 177.

2. 83000 ÷ 55.
3. 24245 ÷ 5555.
4. 1096234 ÷ 4355.
5. 2020005 ÷ 50055.
6. 341220 ÷ 2095.
7. 12156555 ÷ 140655.

179. When the divisor is an aliquot part of some higher unit.

1. Divide 260 by $3\frac{1}{3}$, and 1950 by 25.

OPERATION.

$$\begin{array}{cc} 26|0 & 19|50 \\ 3 & 4 \\ \hline 78 & 78 \end{array}$$

SOLUTION. — Since $3\frac{1}{3}$ is $\frac{1}{3}$ of 10 (§ 140), the next higher unit, we divide 260 by 10; and having used 3 times our true divisor, we obtain only $\frac{1}{3}$ of our true quotient. Multiplying the result, 26, by 3, we have 78, the true quotient.

Since 25 is $\frac{1}{4}$ of 100, the next higher unit, we divide 1950 by 100; and having used 4 times our true divisor, the result, 19.5, is only $\frac{1}{4}$ of our true quotient. Multiplying 19.5 by 4, we have 78, the true quotient.

RULE. — I. *Divide the given dividend by a unit of the order next higher than the divisor, by cutting off figures from the right.*

II. *Take as many times this quotient as the divisor is contained times in the next higher unit.*

2. 63475 ÷ 25.
3. 7856 ÷ 125.
4. 516 ÷ $333\frac{1}{3}$.
5. 16.7324 ÷ $12\frac{1}{2}$.
6. 1748 ÷ $14\frac{2}{7}$.
7. 576.34 ÷ $16\frac{2}{3}$.

180. When the right-hand figure or figures of the divisor are an aliquot part of 10, 100, 1000, etc.

1. Divide $2692366\frac{2}{3}$ by $1233\frac{1}{3}$.

OPERATION.

$$\begin{array}{r|l}
1233\frac{1}{3}) & 2692366\frac{2}{3} \\
3 & 3 \\
\hline
37|00\,) & 80771|00\,(2183\ \textit{Ans.} \\
 & 67 \\
 & 307 \\
 & 111
\end{array}$$

SOLUTION. — Since $33\frac{1}{3}$ is $\frac{1}{3}$ of 100, we multiply both dividend and divisor by 3 (§ 169, III), and we obtain a divisor the component factors of which are 100 and 37. We then divide by contracted division (§ 173).

2. Divide 601387 by 1875.

OPERATION.

$$\begin{array}{r|l}
1875) & 601387 \\
4 & 4 \\
\hline
7500) & 2405548 \\
4 & 4 \\
\hline
3|0000) & 962|2192 \\
\hline
 & 320\frac{1387}{1875}\ \textit{Ans.}
\end{array}$$

SOLUTION. — Multiplying both dividend and divisor by 4, we obtain a new divisor, 7500, having 2 ciphers on the right of it. Multiplying again by 4, we obtain a new divisor, 30000, having 4 ciphers on the right. Then dividing the new dividend by the new divisor, we obtain 320 for a quotient, and 22192 for a remainder. As this remainder is a part of the new dividend, it must be $4 \times 4 = 16$ times the true remainder; we therefore divide it by 16, and write the result over the given divisor, 1875, and annex the fraction thus formed to the integers of the quotient.

RULE. — I. *Multiply both dividend and divisor by a number or numbers that will produce for a new divisor a number ending in a cipher or ciphers.*

II. *Divide the new dividend by the new divisor.*

NOTE. — If the divisor is a whole number, the multiplier will be 2, 4, 5, or 8, or some multiple of one of these numbers.

Divide:

3. 64375 by 2575.
4. 76394 by 3625.
5. 7325 by $433\frac{1}{3}$.
6. 5736 by 43125.
7. 42.75 by $566\frac{2}{3}$.
8. 24409375 by 21875.

Find the value of:

9. $6225 \div 66\frac{2}{3}$.
10. $134.24 \div 187\frac{1}{2}$.
11. $22250 \div 125$.
12. $234.95 \div 175$.
13. $100000 \div 111\frac{1}{9}$.
14. $2413.15 \div 625$.

PROOFS OF DIVISION.

181. To prove division by multiplication.

The dividend is a product of which the divisor and quotient are factors.

RULE. — *Multiply the quotient by the divisor, and to the product add the remainder, if any. If the result is equal to the dividend, the work is correct.*

182. To prove division by casting out 9's.

1. Prove by excess of 9's that $2075 \div 22 = 94$ with a remainder of 7.

OPERATION.

$$\begin{array}{ll} & 22)\,2075\,(94, \text{ exc. } 4 \\ \text{exc. } 4 & \quad\;\; \underline{\quad 7} \\ & \quad\;\; 2068, \text{ exc. } 7. \\ & 4 \times 4 = 16, \text{ exc. } 7 \text{ in prod.} \end{array}$$

SOLUTION. — The excess of 9's in the divisor is 4 (§ 77), and in the quotient 4; the excess in their product 16, is 7. After subtracting the remainder from the dividend, we have 2068, in which the excess of 9's is also 7; hence the answer is probably correct.

RULE. — *Find the excess of 9's in the quotient and divisor and take their product. Cast out the 9's from this product and compare the result with the excess of 9's in the dividend after the remainder, if any, has been subtracted.*

183. To prove division by casting out 11's.

1. Prove by excess of 11's that $2075 \div 22 = 94$ with a remainder of 7.

OPERATION.

$$\begin{array}{ll} & 22)\,2075\,(94, \text{ exc.} = 4 + (11) - 9 = 6. \\ \text{exc. } 0 & \quad\;\; \underline{\quad 7} \\ & \quad\;\; 2068, \; 8 - 8 = 0, \text{ exc.} \\ 0 \times 6 = 0, \text{ exc. in prod.} & \end{array}$$

SOLUTION. — The excess of 11's in the divisor is 0 (§ 78), and in the quotient 6; their product is 0. Subtracting 7 from 2075, the excess of 11's in the remaining dividend is 0, hence the answer is probably correct.

RULE. — *Find the excess of 11's in the quotient and divisor and take their product. Cast out the 11's from this product and compare the result with the excess of 11's in the dividend after the remainder, if any, has been subtracted.*

184. To prove division by summing up the digits.

1. Prove, by summing up the digits, that $2075 \div 22 = 94$, with a remainder of 7.

OPERATION.

$$\begin{array}{l} 22)\,2075\,(94 \\ \;4 \qquad \underline{\;\;7\;}\; 13,\ 4 \\ \qquad 2068 \\ 4 \times 4 = 16, \text{ final digit, } 7 \\ \qquad 2068, \text{ final digit, } 7 \end{array}$$

SOLUTION.—Summing up the digits in the divisor, the result is 4 (§ 79), and in the quotient, 4; their product, 16, gives the final digit 7. Subtracting the remainder, 7, from the divisor, we have 2068, and the final digit of this is also 7, hence the answer is probably correct.

RULE.—*Sum up the quotient and divisor, and find the final digit of the product. Compare with the final digit of the dividend after the remainder, if any, has been subtracted.*

Examples Combining the Preceding Rules.

185. 1. How many barrels of flour at $6 a barrel will pay for 24 tons of coal at $4 a ton, and 36 tons at $6 a ton?

2. A farmer exchanged his farm of 240 acres for another of 150 acres, valued at $40 an acre. What was the value per acre of the farm he sold?

3. A farmer sold some wheat for $600, and corn and oats to the amount of $750. With the proceeds he bought 120 head of sheep at $3 a head, one pair of oxen for $90, and 25 acres of land for the remainder. How much did the land cost him per acre?

4. The product of three numbers is 107100; one of the numbers is 42, and another is 34. What is the third number?

5. A mechanic earns $60 a month, but his necessary expenses are $42 a month. How long will it take him to pay for a farm of 50 acres worth $36 an acre?

6. What number is that which being divided by 45, the quotient increased by $7^2 + 1$, the sum diminished by the difference between 28 and 16, the remainder multiplied by 6, and the product divided by 24, the quotient will be 12?

7. If 532 is the quotient of a number, and the dividend is 251104, what is the divisor?

8. Of what number is 3042 both divisor and quotient?

9. What must the number be which, divided by 453, will give the quotient 307, and the remainder 109?

10. A man at the age of 33 insured his life in favor of his wife for $ 75000, paying $ 25 a year for every $ 1000 of insurance. How much insurance did he pay each year? He died at the age of 60. How much did the sum his wife received exceed the amount the man had paid for insurance?

11. If a tax of $ 5 is levied in a town on every $ 1000 worth of real estate, how much will a man be taxed who owns real estate worth $ 275500?

12. What number will leave a remainder of 10 when subtracted 162 times from 32572?

13. A farmer bought an equal number of sheep and hogs for $ 1276. He paid $ 4 a head for the sheep, and $ 7 a head for the hogs. What was the whole number purchased, and how much was the difference in the total cost of each kind of animal?

14. What number will have a remainder of 36 if subtracted 25 times from 19336?

15. A drover who bought a certain number of cattle for $ 9800, sold some of them for $ 7680, at $ 64 a head, and gained on those he sold $ 960. How much a head did he gain, and how many did he buy?

16. A house and lot valued at $ 1200, and 6 horses at $ 95 each, were exchanged for 30 acres of land. At how much was the land valued per acre?

17. A and B were two candidates for an office. A received 205623 votes more than B, and the total number of votes cast for both was 575249. How many votes did each receive?

18. Alaska has an area of 577390 square miles and Delaware 2050 square miles. How many states the size of Delaware could be made out of Alaska, and how many square miles would remain?

19. The product of two numbers is 55138400. One of the numbers is 31400; what is the other?

20. If 16 men can perform a piece of work in 36 days, in how many days can they perform the same work with the assistance of 8 more men?

21. A dealer bought 275 lamps for $ 1650, and sold 186 at $ 9 each, and the remainder at cost. How much was gained by the transaction?

22. A grocer wishes to put 840 pounds of tea into three kinds of boxes, containing respectively, 5, 10, and 15 pounds, using the same number of boxes of each kind. How many boxes can he fill?

23. Since light moves 192000 miles a second, the difference of time between an electric shock and a flash of lightning is too small to be taken into account; but sound moves only 1142 feet a second. If you hear a thunderclap 8 seconds after you see a flash of lightning, how many miles distant is the shock, reckoning 5280 ft. to the mile?

24. A coal dealer paid $ 965 for some coal. He sold 160 tons for $ 5 a ton, and then the remainder cost him but $ 3 a ton. How many tons did he buy?

25. A dealer in horses paid $ 7560 for a certain number, and sold a part of them for $ 3825, at $ 85 each. By doing so, he lost $ 5 a head. For how much a head must he sell the remainder, to gain $ 945 on the whole?

26. Mr. Clarke bought a western farm for $ 22360, and after expending $ 1720 in improvements upon it, he sold one half of it for $ 15480, at $ 18 per acre. How many acres of land did he purchase, and what was the total cost per acre?

27. If an insurance company has 45 clerks in a room 49 ft. by 110 ft., what is the average amount of floor space for each clerk?

28. If the rent of a store is $ 15000 a year, what is the average rental per day, counting 365 days to the year?

29. The product of three numbers is 12000. The first is 25, and the product of the first by the second 2000. What are the three numbers?

AVERAGE.

186. An **Average** or **Mean** of two or more quantities is their sum divided by the number of quantities added.

Averaging consists of two distinct processes, as follows:

1. The process of finding the mean or average value of two or more things of different given values.

2. The process of finding the proportional quantities of articles at given prices or values, to be used to make a combination of a given average value.

NOTE. — The second process will be explained on p. 295, under "Alligation Alternate."

Examples.

187. To find the average value of things of different value.

1. A miller mixes 30 bushels of rye worth \$.75 a bushel, and 20 bushels of corn worth \$.55 a bushel, with 10 bushels of wheat worth \$.85 a bushel. What is the value of a bushel of the mixture?

OPERATION.

$$\begin{array}{r}
\$.75 \times 30 = \$\,22.50 \\
\$.55 \times 20 = \quad 11.00 \\
\$.85 \times \underline{10} = \quad \underline{8.50} \\
60 \;\big)\,\underline{\$\,42.00} \\
\$.70 \;\; \textit{Ans.}
\end{array}$$

SOLUTION. — Since 30 bushels of rye at \$.75 a bushel are worth \$22.50, and 20 bushels of corn at \$.55 a bushel are worth \$11, and 10 bushels of wheat at \$.85 a bushel are worth \$8.50, therefore the entire mixture, consisting of 60 bushels, is worth \$42, and one bushel is worth $\frac{1}{60}$ of \$42, or \$42 ÷ 60 = \$.70.

RULE. — *Divide the entire cost or value of the ingredients or quantities by their sum.*

2. A student received the following returns from an examination: arithmetic, 98% history, 95%; geography, 100%; grammar, 100%; drawing, 82%. What was his average per cent?

SOLUTION. —

$$98\% + 95\% + 100\% + 100\% + 82\% = 475\%;\ 475\% \div 5 = 95\%.$$

3. If a grocer mixes 8 pounds of tea worth \$.60 a pound with 6 pounds at \$.70 a pound, 2 pounds at \$1.10, and 4 pounds at \$1.20, what is 1 pound of the mixture worth?

4. A grocer mixed 10 pounds of sugar at \$.04 a pound, with 12 pounds @ \$.04½ and 16 pounds @ \$.05½. He sold the mixture @ \$.06 a pound. Did he gain or lose, and how much?

5. On a certain day the thermometer registered the following averages: from 6 to 9 A.M., 64°; from 9 to 12, 74°; from 12 to 3 P.M., 84°; and from 3 to 6, 70°. What was the mean temperature of the day?

6. The register of attendance of a certain school for one week was as follows: Monday, 1050; Tuesday, 1075; Wednesday, 985; Thursday, 1010; Friday, 1095. What was the average daily attendance for that week?

7. On Sunday, June 5, the temperature in New York City averaged 75°, on the following Monday 79°, on Tuesday 83°, on Wednesday 85°, on Thursday 80°, on Friday 95°, on Saturday 90°. Find the average temperature for the week.

8. The expense account of an agent who was on the road for five days was \$6.95 the first day, \$7.50 the second day, \$5.00 the third day, \$8.75 the fourth day, and \$10.00 the fifth day. What were his average daily expenses?

9. The quotations on oats in the daily paper for six successive days were as follows: 38¢ per bushel, 35¢, 33¢, 31¢, 34¢, 35¢. Find the average price per bushel for the week.

10. If a man buys 5 horses, paying for them respectively \$350, \$275, \$150, \$260.50, and \$375.50, what is the average price he pays per horse?

SIGNS OF AGGREGATION.

188. When we wish to indicate that several quantities are to be subjected to the same operation, we inclose them in one of the *signs of aggregation:* in parentheses ( ), in braces { }, in brackets [ ], or under a vinculum ——. Such quantities are said to be in *parenthesis* (§ 32, note).

A quantity in parenthesis must be changed to a single quantity by performing the operations indicated. When there is one parenthesis within another, the inside one should be first removed, and then the next, until none remains. When a number is placed before a parenthesis, brace, or bracket without any sign, the sign × is understood. Thus, $5(3+2)$ means $5 \times (3+2)$.

Precedence is given to the signs × and ÷ over the signs + and − ; hence the operations of multiplication and division should always be performed before addition and subtraction.

Examples.

189. 1. Find the value of

$$15[3 + 5\{8 + (9 + 6) \div 5 + (12 \times 3) + \overline{15 - 3}\} + 10 - 8]$$

OPERATION.

I. $15[3 + 5\{8 + (9 + 6) \div 5 + (12 \times 3) + \overline{15 - 3}\}$
$+ 10 - 8] =$

II. $15[3 + 5\{8 + 3 + 36 + 12\} + 10 - 8] =$

III. $15[3 + 295 + 10 - 8] =$

IV. $15 \times 300 = 4500$ *Ans.*

Solution. — Removing the inner parentheses ( ) and vinculum by performing the indicated operations, we have II. Removing the braces { }, we have III. Removing the brackets [ ], we have IV., and multiplying 300 by 15 as indicated, we have the answer, 4500.

2. Divide $648 \times (3^2 \times 2^3) \div 9 - (2910 \div 15)$ by $2863 \div \{(4375 \div 175) \times 4^2 + 3^2\}$.

OPERATION.

$$648 \times (3^2 \times 2^3) \div 9 - (2910 \div 15) =$$
$$648 \times (9 \times 8) \div 9 - 194 =$$
$$648 \times 8 - 194 = 4990.$$

$$2863 \div \{(4375 \div 175) \times 4^2 + 3^2\} =$$
$$2863 \div \{25 \times 16 + 9\} =$$
$$2863 \div 409 = 7.$$

$$4990 \div 7 = 712\tfrac{6}{7} \textit{ Ans.}$$

RULE.—*Remove all the expressions in parenthesis by performing the operations indicated, beginning with the inner parenthesis. The answer to the last operation indicated will be the value of the expression.*

Find the value of:

3. $5 \times [13 + 2(3 + 4 \times 6) + 5]$.
4. $25 \times (6 \times 3) \times 4 - (9 \times 8 + 90)$.
5. $\{200 - \overline{8 \times 8} + \overline{3 \times 9} - 8\} \div 5$.
6. $8 \times (96 - 26) \times (5 \times 6) - 13 \times 30(5 \times 4)$.
7. $9 \times [3 + \overline{16 + 5} \div 3 + \{3 + \overline{44 \times 5} + \overline{18 \div 6}\} + 22 - 15] \times 8$.
8. $10 \times \{16 - 4 + 3(2 + 8 - 2) + 3 \times 6(4 \div 2) + 8\} \div 10$.
9. $8 \times [3 + \overline{13 + 2} \div 3 + \{\overline{25 \div 5} + \overline{18 \times 6}\} \div 4 + \overline{28 \div 7}]$.
10. Multiply $675 - (77 + 56)$ by $(3 \times 156) - (214 - 28)$.
11. Multiply $98 + 6 \times (37 + 50)$ by $(64 - 50) \times 5 - 10$.
12. What is the product of:
$\{(14 \times 25) - (9 \times 36) + 4324\}$ by $\{(280 - 112) + (376 + 42) \times 4\}$?
13. Divide $450 + (24 - 12) \times 5$ by $(90 \div 6) + (3 \times 11) - 18$.
14. Divide $25 \times (8^2 + 11) \div 5$ by $(225 \div 15) + (12 + 30) \div 7 + (12 + 4) \div 4$.

PROBLEMS IN SIMPLE INTEGRAL NUMBERS.

190. The four operations that have now been considered, viz., Addition, Subtraction, Multiplication, and Division, are all the operations that can be performed upon numbers, hence they are called the **Fundamental Rules.**

In all cases, the numbers operated upon and the results obtained, sustain to each other the relation of a whole to its parts.

I. In **Addition**, the numbers added are the parts, and the sum or amount is the whole.

II. In **Subtraction**, the subtrahend and remainder are the parts, and the minuend is the whole.

III. In **Multiplication**, the multiplicand denotes the value of one part, the multiplier the number of parts, and the product the total value of the whole number of parts.

IV. In **Division**, the dividend denotes the total value of the whole number of parts, the divisor the value of one part, and the quotient the number of parts; or the divisor the number of parts, and the quotient the value of one part.

Every example that can possibly occur in Arithmetic, and every business computation requiring an arithmetical operation, can be classed under one or more of the four Fundamental Rules, as follows:

I. **Cases requiring addition.**

There may be given	*To find*
1. The parts,	the whole, or the sum total.
2. The less of two numbers and their difference, or the subtrahend and remainder,	the greater number or the minuend.

II. **Cases requiring subtraction.**

There may be given	*To find*
1. The sum of two numbers and one of them,	the other.
2. The greater and the less of two numbers, or the minuend and subtrahend,	the difference or remainder.

III. **Cases requiring multiplication.**

There may be given	*To find*
1. Two numbers,	their product.
2. Any number of factors,	their continued product.
3. The divisor and quotient,	the dividend.

IV. **Cases requiring division.**

There may be given	*To find*
1. The dividend and divisor,	the quotient.
2. The dividend and quotient,	the divisor.
3. The product and one of two factors,	the other factor.
4. The continued product of several factors, and the product of all but one factor,	that one factor.

191. Illustrate the following problems by original examples:

1. Given, several numbers, to find their sum.

2. Given, the sum of several numbers and all of them but one, to find that one.

3. Given, the parts, to find the whole.

4. Given, the whole and all the parts but one, to find that one.

5. Given, two numbers, to find their difference.

6. Given, the greater of two numbers and their difference, to find the less number.

7. Given, the less of two numbers and their difference, to find the greater number.

8. Given, the minuend and the subtrahend, to find the remainder.

9. Given, the minuend and the remainder, to find the subtrahend.

10. Given, the subtrahend and the remainder, to find the minuend.

11. Given, two or more numbers, to find their product.

12. Given, the product and one of two factors, to find the other factor.

13. Given, the continued product of several factors and all the factors but one, to find that factor.

14. Given, the factors, to find their product.

15. Given, the multiplicand and the multiplier, to find the product.

16. Given, the product and the multiplicand, to find the multiplier.

17. Given, the product and the multiplier to find the multiplicand.

18. Given, two numbers, to find their quotients.

19. Given, the divisor and the dividend, to find the quotient.

20. Given, the divisor and the quotient, to find the dividend.

21. Given, the dividend and the quotient, to find the divisor.

22. Given, the divisor, the quotient, and the remainder, to find the dividend.

23. Given, the dividend, the quotient, and the remainder, to find the divisor.

24. Given, the final quotient of a continued division and the several divisors, to find the dividend.

25. Given, the final quotient of a continued division, the first dividend, and all the divisors but one, to find that divisor.

26. Given, the dividend and several divisors of a continued division, to find the quotient.

27. Given, two or more sets of numbers, to find the difference of their sums.

28. Given, two or more sets of factors, to find the sum of their products.

29. Given, one or more sets of factors and one or more numbers, to find the sum of the products and the given numbers.

30. Given, two or more sets of factors, to find the difference of their products.

31. Given, one or more sets of factors and one or more numbers, to find the sum of the products and the given number or numbers.

32. Given, two or more sets of factors and two or more other sets of factors, to find the difference of the sums of the products of the former and latter.

33. Given, the sum and the difference of two numbers, to find the numbers.

SOLUTION. — If the difference of two unequal numbers is added to the less number, the sum will be equal to the greater; and if this sum is added to the greater number, the result will be twice the greater number. But this result is obtained by adding the greater number to the less *plus* the difference, or by adding the sum of the two numbers to their difference. Hence the sum of two numbers *plus* their difference equals twice the greater number. Thus the sum of 175 and 125 is 300, and their difference is 50. The less number 125 + the difference 50 = 175, the greater number. Adding the greater number, the result is 175 + 125 + 50 = 175 + 175, or twice the greater number. But since 175 + 125 = the sum 300, hence 300 + 50 = 2 times the greater number.

Again, if the difference of two numbers is subtracted from the greater number, the remainder will be equal to the less number; and if this remainder is added to the less number, the result will be twice the less number. But this result is obtained by subtracting the difference from the greater number *plus* the less, or from the sum. Hence the sum of two numbers *minus* their difference equals twice the less number. Thus, 175 − 50 = 125, the less number. Adding 125, the result is 125 + 175 − 50 = 125 + 125, or twice the less number. Hence 300 − 50 = twice the less number.

PRINCIPLES. — I. *The sum of two numbers* PLUS *their difference is equal to twice the greater number.*

II. *The sum of two numbers* MINUS *their difference is equal to twice the less number.*

PROBLEMS INVOLVING PRICE, QUANTITY, AND COST.

192. In transactions involving money the three elements of price, quantity, and cost must be considered.

193. The **Price** is the value of a unit of any article; the **Quantity** is the number of units taken, and the **Cost** is the value of the whole quantity.

Hence the cost is a product whose factors are the price and quantity. The cost may also be considered as a dividend, with the quantity as a divisor and the price as a quotient, or with the price as a divisor and the quantity as a quotient.

$$\begin{aligned} \text{Cost} &= \text{price} \times \text{quantity}. \\ \text{Price} &= \text{cost} \div \text{quantity}. \\ \text{Quantity} &= \text{cost} \div \text{price}. \end{aligned}$$

194. Given, the price and the quantity, to find the cost.

SOLUTION. — The cost of 3 units must be three times the price of 1 unit; of 8 units, eight times the price of 1 unit, etc.

RULE. — *Multiply the price of one by the quantity.*

195. Given, the cost and the quantity, to find the price.

SOLUTION. — By § 194 the cost is the product of the price multiplied by the quantity. Having the cost, which is a product, and the quantity, which is one of two factors, we have the product and one of two factors given, to find the other factor.

RULE. — *Divide the cost by the quantity.*

196. Given, the price and the cost, to find the quantity.

SOLUTION. — Reasoning as in § 195, we find that the cost is the product of two factors, and the price is one of the factors.

RULE. — *Divide the cost by the price.*

197. Given, the quantity, and the price of 100 or 1000, to find the cost.

SOLUTION. — If the price of 100 units is multiplied by the number of units in a given quantity, the product will be 100 times the required result, because the multiplier used is 100 times the true multiplier. For a similar reason, if the price of 1000 units is multiplied by the number of units in a given quantity, the product will be 1000 times the required result. These errors can be corrected in two ways:

1. By dividing the product by 100 or 1000, as the case may be.
2. By reducing the given quantity to hundreds and decimals of a hundred, or to thousands and decimals of a thousand.

RULE. — *Multiply the price by the quantity reduced to hundreds and decimals of a hundred, or to thousands and decimals of a thousand.*

NOTE. — In business transactions the Roman numerals C and M are commonly used to indicate hundreds and thousands, where the price is by the 100 or 1000.

198. To find the cost of articles sold by the ton of 2000 pounds.

SOLUTION. — If the price of 1 ton or 2000 pounds is divided by 2, the quotient will be the price of ½ ton or 1000 pounds. We then have the quantity and the price of 1000 to find the cost.

RULE. — *Divide the price of* 1 *ton by* 2, *and multiply the quotient by the number of pounds expressed as thousandths.*

Examples.

199. 1. What will be the cost of 187 barrels of salt, at $1.32 a barrel?

2. How much will 5 firkins of butter cost, each containing 70 pounds, at $.20 a pound?

3. If the board of a family is $547.50 for 1 year, how much is it per day?

4. What is the value of 140 sacks of guano, each sack containing 162.5 pounds, at $17 a ton?

5. What will be the cost of 3240 peach trees, at $16 per hundred?

6. At $66.44 a ton, what will be the cost of 842 tons of railroad iron?

7. A man purchased a farm of 325 acres for $10660. How much did it cost per acre?

8. What will be the cost of 840 feet of plank, at $2 per C; and 1262 pickets, at $12 per M?

9. If I pay $6.00 for silk costing $.75 a yard, how many yards do I buy?

10. What is the price of wheat per bushel, when 24 bushels cost $20.88?

DIVISORS, FACTORS, AND MULTIPLES.

EXACT DIVISORS.

200. An **Exact Divisor** of a number is one that will divide it without a remainder. Since division is the reverse of multiplication, it follows that all the exact divisors of a number are factors of that number, and that all its factors are exact divisors.

NOTES. — 1. Every number is divisible by itself and unity; but the number itself and unity are not generally considered as factors, or exact divisors of the number.

2. An exact divisor of a number is sometimes called a *measure* or a *submultiple* of the number.

201. An **Even Number** is a number of which 2 is an exact divisor; as 2, 4, 6, or 8.

202. An **Odd Number** is a number of which 2 is not an exact divisor; as 1, 3, 5, 7, or 9.

203. A **Perfect Number** is one that is equal to the sum of all its factors plus 1; as $6 = 3 + 2 + 1$, or $28 = 14 + 7 + 4 + 2 + 1$.

NOTE. — The only perfect numbers known are:
6, 28, 496, 8128, 33550336, 8589869056, 137438691328, 2305843008139952128, 2417851639228158837784576, 9903520314282971830448816128.

204. An **Imperfect Number** is one that is not equal to the sum of all its factors plus 1; as 12, which is not equal to $6 + 4 + 3 + 2 + 1$.

205. An **Abundant Number** is one which is less than the sum of all its factors plus 1; as 18, which is less than $9 + 6 + 3 + 2 + 1$.

206. A **Defective Number** is one which is greater than the sum of all its factors plus 1; as 27, which is greater than 9 + 3 + 1.

207. To show the nature of exact division, and furnish tests of divisibility, observe that if we begin with any number, as 4, and take once 4, two times 4, three times 4, four times 4, and so on indefinitely, forming the series 4, 8, 12, 16, etc., we shall have all the numbers that are divisible by 4; and from the manner of forming this series, it is evident that:

1. The *product* of any one number of the series by any integral number whatever, will contain 4 an exact number of times.
2. The *sum* of any two numbers of the series will contain 4 an exact number of times.
3. The *difference* of any two will contain 4 an exact number of times.

PRINCIPLES.—I. *Any number which will exactly divide one of two numbers will divide their product.*

II. *Any number which will exactly divide each of two numbers will divide their sum.*

III. *Any number which will exactly divide each of two numbers will divide their difference.*

208. From these principles we derive the following facts or properties of numbers:

I. *Any number terminating with* 0, 00, 000, *etc., is divisible by* 10, 100, 1000, *etc., or by any factor of* 10, 100, *or* 1000.

For, by cutting off the cipher or ciphers, the number will be divided by 10, 100, or 1000, etc., without a remainder (§ 176), and a number of which 10, 100, or 1000, etc., is a factor, will contain any factor of 10, 100, or 1000, etc. (§ 207, I).

II. *A number is divisible by* 2 *if the number expressed by its right-hand figure is even or divisible by* 2.

For, the part at the left of the units' place, taken alone, with its local value, is a number which terminates with a cipher, and is divisible by 2, because 2 is a factor of 10; and if both parts, taken separately, with their local values, are divisible by 2, their sum, which is the entire number, is divisible by 2 (§ 207, II).

NOTE.—Hence, all numbers terminating with 0, 2, 4, 6, or 8, are *even*, and all numbers terminating with 1, 3, 5, 7, or 9, are *odd*.

III. *A number is divisible by* 4 *if the number expressed by its two right-hand figures is divisible by* 4.

For, the part at the left of the tens' place, taken alone, with its local value, is a number which terminates with two ciphers, and is divisible by 4, because 4 is a factor of 100; and if both parts, taken separately, with their local values, are divisible by 4, their sum, which is the entire number, is divisible by 4 (§ 207, II).

IV. *A number is divisible by* 8 *if the number expressed by its three right-hand figures is divisible by* 8.

For, the part at the left of the hundreds' place, taken alone, with its local value, is a number which terminates with three ciphers, and is divisible by 8, because 8 is a factor of 1000, and if both parts, taken separately, with their local values, are divisible by 8, their sum, or the entire number, is divisible by 8 (§ 207, II).

V. *A number is divisible by* 16 *if the number expressed by its four right-hand figures is divisible by* 16.

For, the part at the left of the thousands' place, taken alone, with its local value, is a number which terminates with four ciphers and is divisible by 16, because 16 is a factor of 10000, and if both parts taken separately with their local value are divisible by 16, their sum or the entire number is divisible by 16 (§ 207, II).

VI. *A number is divisible by any power of* 2, *if as many of the right-hand terms of the number as are equal to the index of the given power are divisible by the given power.*

For, as 2 is a factor of 10, any power of 2 is a factor of the corresponding power of 10, or of a unit of an order one higher than is indicated by the index of the given power of 2; and if both parts of a number, taken separately, with their local values, are divisible by a power of 2, their sum, or the entire number, is divisible by the same power of 2 (§ 207, II).

VII. *A number is divisible by* 5 *if its right-hand figure is* 0 *or* 5.

For, if a number terminates with a cipher, it is divisible by 5, because 5 is a factor of 10, and if it terminates with 5, both parts, the units and the figures at the left of units, taken separately, with their local values, are divisible by 5, and consequently their sum, or the entire number, is divisible by 5 (§ 207, II).

VIII. *A number is divisible by* 25 *if the number expressed by its two right-hand figures is divisible by* 25.

For, the part at the left of the tens' place, taken with its local value, is a number terminating with two ciphers, and is divisible by 25, because 25 is a factor of 100, and if both parts, taken separately, with their local values, are divisible by 25, their sum, or the entire number, is divisible by 25 (§ 207, II).

IX. *A number is divisible by any power of* 5, *if as many of the right-hand terms of the number as are equal to the index of the given power are divisible by the given power.*

For, as 5 is a factor of 10, any power of 5 is a factor of the corresponding power of 10, or of a unit of an order one higher than is indicated by the index of the given power of 5; and if both parts of a number, taken separately, with their local values, are divisible by a power of 5, their sum, or the entire number, is divisible by the same power of 5 (§ 207, II).

X. *A number is divisible by* 9 *if the sum of its digits is divisible by* 9.

For, if any number, as 7245, is separated into its parts, $7000 + 200 + 40 + 5$, and each part is divided by 9, the several remainders will be the digits 7, 2, 4, and 5, respectively (§ 77); hence, if the sum of these digits, or remainders, is 9, or an exact number of 9's, the entire number must contain an exact number of 9's, and will therefore be divisible by 9. From this it follows also that every number is an exact number of 9's plus the sum of its digits (§ 77).

XI. *A number is divisible by* 3 *if the sum of its digits is divisible by* 3.

For every number is an exact number of 9's plus the sum of its digits (X and § 77). The part which is an exact number of 9's is divisible by 3; hence if the remainder, which is the sum of its digits, is divisible by 3 the whole number is divisible by 3. Thus $357 = (38 \times 9) + 3 + 5 + 7$. 38×9 or $38 \times 3 \times 3$ is divisible by 3, and 15 is divisible by 3, hence 357 is divisible by 3.

XII. *An even number is divisible by* 6 *if the sum of its digits is divisible by* 3.

For, the number is divisible by 3 because the sum of its digits is divisible by 3, and it is divisible by 2 because it is even; hence, it is divisible by 3×2, or 6 (§ 172, I).

XIII. *A number is divisible by* 11 *when the sum of the digits in the odd place, minus the sum of the digits in the even place, is divisible by* 11, *or is* 0.

For, even powers of 10 are multiples of 11 plus 1 (100, the square of $10 = 9 \times 11 + 1$; 1000, the fourth power $= 909 \times 11 + 1$, etc.); odd

powers of 10 are multiples of 11 − 1 (10, the first power = 11 − 1; 1000, the third power = 91 × 11 − 1, etc.). Therefore, if a number expressed by a digit in an odd place is divided by 11, the remainder will be equal to that digit; and a number expressed by a digit in an even place will lack that digit of being a multiple of 11.

Thus, in 3839:

$$\begin{aligned} 3000 &= 273 \times 11 - 3. \\ 800 &= 72 \times 11 + 8. \\ 30 &= 3 \times 11 - 3. \\ 9 &= 0 \times 11 + 9. \end{aligned}$$

$$8 + 9 - 3 - 3 = 17 - 6 = 11.$$

Hence, if a number is divided by 11, the remainder will be the same as if the sum of the digits in the odd places (increased by a multiple of 11 if necessary), minus the sum of the digits in the even place, were divided by 11. (See § 78.)

XIV. *A number is divisible by a composite number, when it is divisible, successively, by all the component factors of the composite number.*

For, dividing any number successively by several factors, is the same as dividing by the product of these factors (§ 172, I).

XV. *An odd number is not divisible by an even number.*

For, the product of any number (odd or even) by an even number is *even;* and, consequently, any composite odd number can contain only odd factors.

XVI. *An even number that is divisible by an odd number is also divisible by twice that odd number.*

For, if any even number is divided by an odd number, the quotient must be even, and divisible by 2; hence, the given even number, being divisible successively by the odd number and 2, will be divisible by their product, or twice the odd number (§ 172, I).

XVII. *An even number is divisible by* 18 *if the sum of its digits is divisible by* 9.

For, the number is divisible by 9 because the sum of its digits is divisible by 9 (X), and since it is even it is divisible by 2 × 9 or 18 (XVI).

XVIII. *An even number is divisible by* 22 *if the sum of the digits in the odd place, minus the sum of the digits in the even place, is divisible by* 11 *or is* 0.

For the number is divisible by 11 (XIII); and since it is even it is divisible by 2; hence it is divisible by 2 × 11 or 22 (XVI).

209. From these principles we summarize the following **tests of divisibility:**

2 is an exact divisor of every even number.
3 is an exact divisor of a number if the sum of its digits is divisible by 3.
4 is an exact divisor of a number if the number expressed by its two right-hand figures is divisible by 4, or if its two right-hand figures are ciphers.
5 is an exact divisor of a number if its right-hand figure is 5 or 0.
6 is an exact divisor of an *even* number if the sum of its digits is divisible by 3.
8 is an exact divisor of a number if the number expressed by its three right-hand figures is divisible by 8, or if its three right-hand figures are ciphers.
9 is an exact divisor of a number if the sum of its digits is divisible by 9.
10 is an exact divisor of every number whose units' term is 0.
11 is an exact divisor of a number if the sum of its digits in the odd places minus the sum of its digits in the even places is divisible by 11 or is 0.
12 is an exact divisor of a number divisible by both 3 and 4.
15 is an exact divisor of a number divisible by both 3 and 5.
16 is an exact divisor of a number if the number expressed by its four right-hand figures is divisible by 16, or if the four right-hand figures are ciphers.
18 is an exact divisor of an *even* number if the sum of its digits is divisible by 9.
20 is an exact divisor of a number divisible by 5 and 4.
22 is an exact divisor of an *even* number divisible by 11.
25 is an exact divisor of a number if the number expressed by its two right-hand figures is divisible by 25, or if its two right-hand figures are ciphers.

210. A **Prime Number** is one that cannot be resolved or separated into two or more integral factors.

NOTE. — Every number must be either prime or composite.

To find all the prime numbers within any given limit, we observe that all even numbers except 2 are composite; hence, the prime numbers must be sought among the odd numbers.

NOTE. — Every prime number except 2 and 5 ends with 1, 3, 7, or 9.

211. If the odd numbers are written in their order, thus, 1, 3, 5, 7, 9, 11, 13, 15, 17, etc., we observe that:

1. Taking every third number after 3, we have 3 times 3, 5 times 3, 7 times 3, and so on; which are the only odd numbers divisible by 3.
2. Taking every fifth number after 5, we have 3 times 5, 5 times 5,

7 times 5, and so on; which are the only odd numbers divisible by 5. And the same will be true of every other number in the series. Hence,

3. If we cancel every third number, counting from 3, no number divisible by 3 will be left; and since 3 times 5 will be canceled, 5 times 5, or 25, will be the least composite number left in the series. Hence,

4. If we cancel every fifth number, counting from 25, no number divisible by 5 will be left; and since 3 times 7, and 5 times 7, will be canceled, 7 times 7, or 49, will be the least composite number left in the series. And thus with all the prime numbers.

Rule. — I. *Write all the odd numbers in their natural order.*

II. *Cancel, or cross out, 3 times 3, or 9, and every third number after it; 5 times 5, or 25, and every fifth number after it; 7 times 7, or 49, and every seventh number after it; and so on, beginning with the second power of each prime number in succession, till the given limit is reached. The numbers remaining, together with the number 2, will be the prime numbers required.*

Notes. — 1. It is unnecessary to count for every ninth number after 9 times 9, for being divisible by 3, they will be found already canceled; the same may be said of any other canceled, or composite number.

2. This method of obtaining a list of the prime numbers was employed by Eratosthenes (born B.C. 276), and is called *Eratosthenes' Sieve.*

TABLE OF PRIME NUMBERS LESS THAN 1000.

1	59	139	233	337	439	[illegible]	653	769	883
2	61	[illegible]	239	347	[illegible]	563	[illegible]	773	887
3	67	151	[illegible]	349	[illegible]	569	661	787	907
5	71	157	251	353	457	571	673	797	911
7	73	[illegible]	257	359	[illegible]	577	677	[illegible]	919
11	79	167	263	367	463	587	[illegible]	811	929
13	83	173	269	373	467	[illegible]	691	821	937
17	89	179	271	379	479	599	701	823	941
19	97	181	277	383	[illegible]	601	[illegible]	827	[illegible]
23	101	191	281	389	491	607	719	829	953
29	103	193	283	397	499	613	727	839	967
31	107	[illegible]	293	401	503	617	733	853	971
37	109	199	307	409	509	619	739	857	977
41	113	211	311	419	521	631	743	859	983
43	127	223	313	421	523	641	751	863	[illegible]
47	131	227	317	431	541	643	757	877	997
53	137	229	331	433	547	647	761	881	

FACTORING.

212. The **Prime Factors** of a number are those numbers whose product is the given number.

213. Factoring is based on the following principles:

PRINCIPLES.—I. *Every prime factor of a number is an exact divisor of that number.*

II. *The only exact divisors of a number are its prime factors, or some combination of its prime factors.*

Examples.

214. To resolve any composite number into its prime factors.

1. What are the prime factors of 114?

OPERATION.

2	114
3	57
19	19

$2 \times 3 \times 19$ *Ans.*

SOLUTION.—Since the given number is even, we divide by 2, and obtain an odd number, 57, for a quotient. We then divide by the prime numbers 3 and 19, successively, and the last quotient is 1. The divisors, 2, 3, and 19, are the prime factors required (§ 213, II).

RULE.—*Divide the given number by any prime factor; divide the quotient by another prime factor, and so continue the division until the quotient is a prime number. The several divisors and the last quotient will be the prime factors required.*

What are the prime factors of:

2. 2150?	4. 6300?	6. 2366?	8. 390625?
3. 2445?	5. 21504?	7. 1000?	9. 999999?

215. Small prime factors can be found by the tests (§ 209) or by trial. Others may be found by the table, on pp. 110, 111.

By prefixing each number in bold-face type in the column of "Numbers," to the several numbers following it in the same division of the column, we shall form all the *composite numbers* less than 10000, and *not divisible* by 2, 3, 5, 7, or 11; the numbers in the columns of "Factors" are the *least prime factors* of the numbers thus formed respectively. Thus, in one of the columns of "Numbers" we find 39, in bold-face type, and below 39, in the same column, is 77, which, annexed to 39, forms 3977, a composite number. The least prime factor of this number is 41, which we find at the right of 77, in the column of "Factors."

FACTOR TABLE.

Numbers.	Factors.	Numbers.	Factors.	Numbers.	Factors.	Numbers.	Factors.	Numbers.	Factors.	Numbers.	Factors.	Numbers.	Factors.	Numbers.	Factors.	Numbers.	Factors.	Numbers.	Factors.	Numbers.	Factors.
1		99	29	11	17	43	29	79	37	41	17	83	17	41	23	09	31	17	53	77	31
69	13	**9**		17	13	57	19	83	13	51	13	97	43	59	17	13	19	27	29	83	71
2		01	17	57	31	61	37	89	19	77	13	**34**		69	53	21	29	47	47	91	29
21	13	23	13	69	13	63	13	91	47	83	19	01	19	87	13	31	61	57	67	**52**	
47	13	43	23	**15**		**20**		**25**		87	29	03	41	93	17	43	43	69	19	07	41
89	17	49	13	01	19	21	43	01	41	93	41	19	13	**39**		51	19	71	13	13	13
99	13	61	31	13	17	33	19	07	23	**30**		27	23	01	47	69	17	77	17	19	17
3		89	23	17	37	41	13	09	13	07	31	31	47	37	31	79	29	**48**		21	23
23	17	**10**		37	29	47	23	33	17	13	23	39	19	53	59	81	13	11	17	39	13
61	19	03	17	41	23	59	29	37	43	29	13	73	23	59	37	87	41	19	61	49	29
77	13	07	19	77	19	71	19	61	13	43	17	81	59	61	17	93	23	41	47	51	59
91	17	27	13	91	37	77	31	67	17	53	43	97	13	73	29	99	53	43	29	63	19
4		37	17	**16**		**21**		73	31	71	37	**35**		77	41	**44**		47	37	67	23
03	13	73	29	33	23	17	29	81	29	77	17	03	31	79	23	27	19	49	13	87	17
37	19	79	13	43	31	19	13	87	13	97	19	23	13	91	13	29	43	53	23	93	67
81	13	81	23	49	17	47	19	99	23	**31**		51	53	**40**		39	23	59	43	**53**	
93	17	**11**		51	13	59	17	**26**		03	29	69	43	09	19	53	61	67	31	11	47
5		21	19	79	23	71	13	03	19	07	13	87	17	31	29	69	41	83	19	17	13
27	17	39	17	81	41	73	41	23	43	27	53	89	37	33	37	71	17	91	67	21	17
29	23	47	31	91	19	83	37	27	37	31	31	99	59	43	13	89	67	97	59	29	73
33	13	57	13	**17**		97	13	41	19	33	13	**36**		61	31	**45**		**49**		39	19
51	19	59	19	03	13	**22**		69	17	39	43	01	13	63	17	11	13	01	13	53	53
59	13	89	29	11	29	01	31	**27**		49	47	11	23	69	13	31	23	13	17	59	23
89	19	**12**		17	17	09	47	01	37	51	23	29	19	87	61	37	13	27	13	63	31
6		07	17	39	37	27	17	43	13	61	29	49	41	97	17	41	19	79	13	71	41
11	13	19	23	51	17	31	23	47	41	73	19	53	13	**41**		53	29	81	17	77	19
29	17	41	17	63	41	49	13	59	31	93	31	67	19	17	23	59	47	97	19	89	17
67	23	47	29	69	29	57	37	71	17	97	23	79	13	21	13	73	17	**50**		**54**	
89	13	61	13	81	13	63	31	73	47	**32**		83	29	41	41	77	23	17	29	29	61
97	17	71	31	**18**		79	43	**28**		11	13	**37**		63	23	79	19	29	47	47	13
7		73	19	07	13	91	29	09	53	33	53	13	47	71	43	89	13	41	71	59	53
03	19	**13**		17	23	**23**		13	29	39	41	21	61	81	37	**46**		53	31	61	43
13	23	13	13	19	17	23	23	31	19	47	17	37	37	83	47	01	43	57	13	73	13
31	17	33	31	29	31	27	13	39	17	63	13	43	19	87	53	07	17	63	61	91	17
67	13	39	13	43	19	29	17	67	47	77	29	49	23	89	59	19	31	69	37	97	23
79	19	43	17	49	43	53	13	69	19	81	17	57	13	99	13	33	41	83	13	**55**	
93	13	49	19	53	17	63	17	73	13	87	19	63	53	**42**		61	59	**51**		13	37
99	17	57	23	91	31	69	23	81	43	93	37	81	19	23	41	67	13	11	19	39	29
8		63	29	**19**		**24**		99	13	**33**		91	17	37	19	81	31	23	47	43	23
17	19	69	37	09	23	07	29	**29**		17	31	99	29	47	31	87	43	29	23	49	31
41	29	87	19	19	19	13	19	11	41	37	47	**38**		67	17	93	13	41	53	61	67
51	23	91	13	21	17	19	41	21	23	41	13	09	13	**43**		99	37	43	37	67	19
71	13	**14**		27	41	49	31	23	37	49	17	11	37	03	13	**47**		49	19	87	37
93	19	03	23	37	13	61	23	29	29	79	31	27	43	07	59	09	17	61	13	97	29

FACTOR TABLE. — *Continued.*

Numbers.	Factors.	Numbers.	Factors.	Numbers.	Factors.	Numbers.	Factors.	Numbers.	Factors.	Numbers.	Factors.	Numbers.	Factors.	Numbers.	Factors.	Numbers.	Factors.	Numbers.	Factors.	Numbers.	Factors.
56		**60**		39	47	51	13	61	53	57	13	23	71	13	47	51	53	53	19	59	13
03	13	01	17	43	17	59	19	67	13	61	47	27	23	17	19	57	17	59	47	71	19
09	71	19	13	63	23	77	13	77	19	63	79	33	29	41	23	73	19	63	59	73	17
11	31	23	19	67	29	87	71	79	29	97	43	47	13	53	79	79	13	69	13	83	23
17	41	31	37	87	13	89	83	89	37	**77**		51	83	71	43	81	83	71	73	**97**	
27	17	49	23	93	43	93	61	91	23	09	13	77	41	73	37	91	17	87	37	01	89
29	13	59	73	97	73	**69**		**73**		29	59	83	59	79	61	**89**		99	17	03	31
33	43	71	13	99	67	01	67	03	67	39	71	**81**		83	17	03	29	**93**		07	17
71	53	77	59	**65**		13	31	13	71	47	61	19	23	89	13	09	59	01	71	27	71
81	13	**61**		09	23	29	13	19	13	51	23	31	47	97	29	17	37	07	41	31	37
99	41	03	17	11	17	31	29	27	17	69	17	37	79	**85**		27	79	13	67	61	43
57		07	31	27	61	43	53	39	41	71	19	43	17	07	47	47	23	22	19	63	13
07	13	09	41	33	47	53	17	61	17	81	31	49	29	09	67	57	13	47	13	73	29
13	29	19	29	39	13	73	19	63	37	83	43	53	31	31	19	59	17	53	47	97	97
23	59	37	17	41	31	89	29	67	53	87	13	59	41	49	83	77	47	67	17	99	41
29	17	57	47	57	79	**70**		73	73	**78**		77	13	51	17	83	13	79	83	**98**	
59	13	61	61	83	29	03	47	79	47	01	29	89	19	57	43	89	89	89	41	09	17
67	73	69	31	93	19	09	43	87	83	07	37	**82**		67	13	93	17	**94**		27	31
71	29	79	37	**66**		31	79	91	19	11	73	01	59	79	23	**90**		07	23	41	13
73	23	87	23	13	17	33	13	97	13	13	13	03	13	87	31	17	71	09	97	47	43
77	53	91	41	17	13	37	31	**74**		31	41	07	29	93	13	19	29	51	13	53	59
58		**62**		23	37	61	23	09	31	37	17	13	43	**86**		47	83	69	17	69	71
09	37	27	13	31	19	67	37	21	41	49	47	27	19	11	79	61	13	81	19	81	41
33	19	33	23	41	29	81	73	23	13	59	29	49	73	21	37	71	47	87	53	93	13
37	13	39	17	47	17	87	19	29	17	71	17	51	37	33	89	73	43	**95**		99	19
91	43	41	79	49	61	93	41	39	43	91	13	57	23	39	53	77	29	03	13	**99**	
93	71	53	13	67	59	97	47	53	29	97	53	79	17	51	41	83	31	09	37	13	23
99	17	83	61	83	41	99	31	63	17	**79**		99	43	53	17	89	61	17	31	17	47
59		89	19	97	37	**71**		71	31	13	41	**83**		71	13	**91**		23	89	37	19
09	19	**63**		**67**		11	13	93	59	21	89	03	19	83	19	01	19	29	13	43	61
11	23	13	59	07	19	23	17	**75**		39	17	21	53	**87**		13	13	53	41	53	37
17	61	19	71	31	53	41	37	01	13	43	13	33	13	11	31	31	23	57	19	59	23
21	31	31	13	39	23	53	23	19	73	57	73	39	31	17	23	39	13	63	73	71	13
33	17	41	17	49	17	57	17	31	17	61	19	41	19	49	13	43	41	71	17	79	17
41	13	71	23	51	43	63	13	43	19	67	31	47	17	59	19	67	89	77	61	83	67
47	19	83	13	57	29	69	67	71	67	69	13	57	61	73	31	69	53	89	43	91	97
59	59	**64**		67	67	71	71	97	71	79	79	59	13	77	67	79	67	93	53	97	13
63	67	01	37	73	13	81	43	**76**		81	23	81	17	91	59	93	29	99	29		
69	47	03	19	99	13	99	23	13	23	91	61	83	83	97	19	97	17	**96**			
77	43	07	43	**68**		**72**		19	19	99	19	99	37	**88**		**92**		07	13		
83	31	09	13	17	17	01	19	27	29	**80**		**84**		01	13	11	61	17	59		
89	53	31	59	21	19	23	31	31	13	03	53	01	31	09	23	17	13	37	23		
93	13	37	41	47	41	41	13	33	17	21	13	11	41	43	37	23	23	41	31		

216. To find the prime factors of a number by the table.

1. Resolve 1849 into its prime factors.

OPERATION.

$1849 \div 43 = 43$

$1849 = 43 \times 43$ *Ans.*

SOLUTION. — Cutting off the two right-hand figures of the given number, we find the other part, 18, in the table, in bold-face type in the third column. and under it, in the same division of the column, we find 49, the figures which were cut off; at the right of 49, in the factor column, we find 43, the least prime factor of the given number. Dividing by 43, we obtain 43, the other factor.

2. Resolve 26877 into its prime factors.

OPERATION.

3	26877
17	8959
17	527
31	31

$3 \times 17 \times 17 \times 31$ *Ans.*

SOLUTION. — We find by trial that the given number is divisible by 3 and the quotient is 8959. By the factor table, we find the least prime factor of this number to be 17; dividing by 17, we have 527 for a quotient. Referring again to the table, we find 17 to be the least factor of 527, and the other factor, 31, is prime.

RULE — I. *Cancel from the given number all factors less than* 13, *and then find the remaining factors by the table.*

II. *If any number less than* 10000 *is not found in the table, and is not divisible by* 2, 3, 5, 7, *or* 11, *it is prime.*

Resolve into their prime factors:

3. 18902. 5. 203566. 7. 893235. 9. 6409.

4. 352002. 6. 59843. 8. 390976. 10. 178296.

217. To find all the exact divisors of a number.

All the prime factors of a number, with all their combinations, will constitute all the exact divisors of that number (§ 213, II).

1. What are all the exact divisors of 360?

OPERATION.

$$360 = 1 \times 2 \times 2 \times 2 \times 3 \times 3 \times 5.$$

Ans.	1,	2,	4,	8	Combinations of 1 and 2.
	3,	6,	12,	24	" " 1 and 2 and 3.
	9,	18,	36,	72	
	5,	10,	20,	40	" " 1 and 2 and 5.
	15,	30,	60,	120	" " 1 and 2 and 3 and 5.
	45,	90,	180,	360	

SOLUTION.—By § 214, we find the prime factors of 360 to be 1, 2, 2, 2, 3, 3, and 5. As 2 occurs three times as a factor, the different combinations of 1 and 2 by which 360 is divisible will be 1, $1 \times 2 = 2$, $1 \times 2 \times 2 = 4$, and $1 \times 2 \times 2 \times 2 = 8$; these we write in the first line. Multiplying the first line by 3 and writing the products in the second line, and the second line by 3, writing the products in the third line, we have in the first, second, and third lines all the different combinations of 1, 2, and 3, by which 360 is divisible. Multiplying the first, second, and third lines by 5, and writing the products of the fourth, fifth, and sixth lines, respectively, we have in the six lines together, every combination of the prime factors by which the given number, 360, is divisible.

2. What are all the exact divisors of 960?

OPERATION.

$$960 = 1 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 3 \times 5.$$

Ans.	1,	2,	4,	8,	16,	32,	64	Combinations of 1 and 2.
	3,	6,	12,	24,	48,	96,	192	" " 1 and 2 and 3.
	5,	10,	20,	40,	80,	160,	320	" " 1 and 2 and 5.
	15,	30,	60,	120,	240,	480,	960	" " 1 and 2 and 3 and 5.

RULE.—I. *Resolve the given number into its prime factors.*

II. *Form a series having* 1 *for the first term, that prime factor which occurs the greatest number of times in the given number for the second term, the square of this factor for the third term, and so on, till a term is reached containing this factor as many times as it occurs in the given number.*

III. *Multiply the numbers in this line by another factor, and these results by the same factor, and so on, as many times as this factor occurs in the given number.*

IV. *Multiply all the combinations now obtained by another factor in continued multiplication, and thus proceed till all the different factors have been used.* ALL *the combinations obtained will be the exact divisors sought.*

Find all the exact divisors of:

3. 36.	**6.** 84.	**9.** 100.	**12.** 420.
4. 60.	**7.** 90.	**10.** 125.	**13.** 365.
5. 120.	**8.** 45.	**11.** 150.	**14.** 1050.

GREATEST COMMON DIVISOR.

218. **A Common Divisor** of two or more numbers is a number that will exactly divide each of them.

219. The **Greatest Common Divisor** of two or more numbers is the greatest number that will exactly divide each of them.

220. **Numbers Prime to each other** are such as have no common divisor.

NOTE. — A common divisor is sometimes called a *common measure;* and the greatest common divisor, the *greatest common measure.*

Examples.

221. **To find the greatest common divisor when the numbers can be readily factored.**

It is evident that if several numbers have a common divisor, they may all be divided by any component factor of this divisor, and the resulting quotients by another component factor, and so on, till all the component factors have been used.

1. What is the greatest common divisor of 28, 140, and 420?

OPERATION.

7	28	140	420
4	4	20	60
	1	5	15

$4 \times 7 = 28$ *Ans.*

SOLUTION. — We readily see that 7 will exactly divide each of the given numbers; and then, 4 will exactly divide each of the resulting quotients. Hence, each of the given numbers can be exactly divided by 7 times 4; and these numbers must be component factors of the greatest common divisor. Now, if there were any other component factor of the greatest common divisor, the quotients, 1, 5, and 15, would be divisible by it. But these quotients are prime to each other; therefore, 7 and 4 are all the component factors of the greatest common divisor sought.

RULE. — I. *Write the numbers in a line, and divide by any factor common to all the numbers.*

II. *Divide the quotients in like manner, and continue the division till a set of quotients is obtained that are prime to each other.*

III. *Multiply all the divisors together, and the product will be the greatest common divisor sought.*

What is the greatest common divisor of:

2. 40, 75, and 100?
3. 18, 30, 36, and 42?
4. 102 153, and 255?
5. 756 and 1575?
6. 182, 364, and 455?
7. 42, 63, 126, and 189?
8. 135, 225, 270, and 315?
9. 2520 and 3240?
10. 1428 and 1092?
11. 1008 and 1036?
12. 84, 126, 210, 252, 294, and 462?
13. 216, 360, 432, 648, and 936?

222. To find the greatest common divisor when the numbers cannot be readily factored.

PRINCIPLES. — I. *An exact divisor divides any number of times its dividend* (§ 207, I).

II. *A common divisor of two numbers is an exact divisor of their sum* (§ 207, II).

III. *A common divisor of two numbers is an exact divisor of their difference* (§ 207, III).

1. What is the greatest common divisor of 527 and 1207?

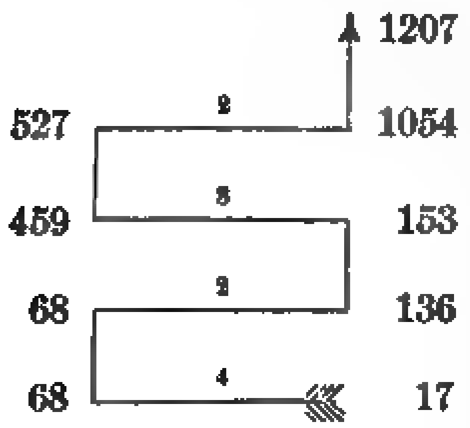
OPERATION.

		1207
527	2	1054
459	3	153
68	2	136
68	4	17 *Ans.*

SOLUTION. — We shall first describe the process, and then examine the reasons for the several steps in the operation. Drawing two vertical lines, we place the greater number on the right, and the less number on the left, one line lower down. We then divide 1207, the greater number, by 527, the less, and write the quotient, 2, between the vertical lines, the product, 1054, opposite the less number and under the greater, and the remainder, 153, below. We next divide 527 by this remainder, writing the quotient, 3, between the verticals, the product, 459, on the left, and the remainder, 68, below. We again divide the last divisor, 153, by 68, and obtain 2 for a quotient, 136 for a product, and 17 for a remainder, all of which we write in the same order as in the former steps. Finally, dividing the last divisor, 68, by the last remainder, 17, we have no remainder, and 17, the last divisor, is the greatest common divisor of the given numbers.

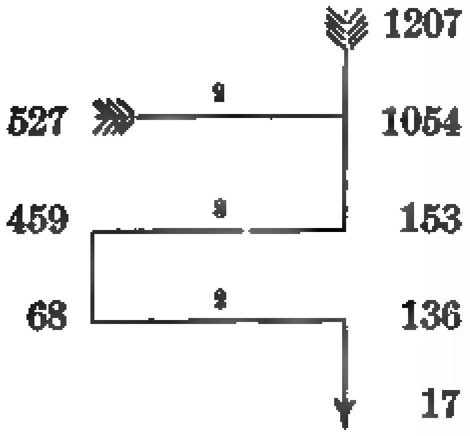
Now, observing that the dividend is always the *sum* of the product and remainder, and that the remainder is always the *difference* of the dividend and product, we first trace the work in the reverse order, as indicated by the arrow line in the diagram on p. 116.

OPERATION.



17 divides 68, as provided by the last division; it will also divide 2 times 68, or 136 (I). Now, as 17 divides both itself and 136, it will divide 153, their sum (II). It will also divide 3 times 153, or 459 (I); and since it is a common divisor of 459 and 68, it must divide their sum, 527, which is one of the given numbers. It will also divide 2 times 527, or 1054 (I); and since it is a common divisor of 1054 and 153, it must divide their sum, 1207, the greater number (II). Hence, 17 is a *common divisor* of the given numbers.

Again, tracing the work in the direct order, as indicated in the following diagram, we know that the *greatest* common divisor, *whatever it may be*, must divide 2 times 527, or 1054 (I). And since it will divide both 1054 and 1207, it must divide their difference, 153 (III). It will also divide 3 times 153, or 459 (I); and as it will divide both 459 and 527, it must divide their difference, 68 (III). It will also divide 2 times 68, or 136 (I); and as it will divide both 136 and 153, it must divide their difference, 17 (III); *hence, it cannot be greater than* 17.



Thus, we have shown, first, that 17 is a *common divisor* of the given numbers, and second, that their greatest common divisor, whatever it may be, cannot be *greater than* 17. Hence it must be 17.

This work may also be indicated by the ordinary form of division:

OPERATION.

```
527)1207(2
    1054
     153)527(3
         459
          68)153(2
             136
              17)68(4
                 68
```

NOTE. — The method in this case is the same as in the other, only the form of indicating it is modified.

RULE.—I. *Divide the greater number by the less.*

II. *If there is a remainder divide the preceding divisor by it, and so continue until there is no remainder; the last divisor will be the greatest common divisor sought.*

III. *If more than two numbers are given, first find the greatest common divisor of two of them, and then of this divisor and one of the remaining numbers, and so on to the last; the last common divisor found will be the greatest common divisor required.*

NOTES.—1. When more than two numbers are given, it is better to begin with the least two.

2. If at any point in the operation a *prime* number occurs as a remainder, it must be a common divisor, or the given numbers have no common divisor.

2. What is the greatest common divisor of 10661 and 12303?

OPERATION.

			12303
	10661	1	10661
	9852	6	1642
Prime	809		

Ans. 1.

What is the greatest common divisor of:

3. 336 and 812?
4. 407 and 1067?
5. 825 and 1372?
6. 2041 and 8476?
7. 3281 and 10778?
8. 22579 and 116939?
9. 49373 and 147731?
10. 18607 and 417979?
11. 1005973 and 4616175?
12. 292, 1022, and 1095?
13. 4718, 6951, and 8876?
14. 141, 799, and 940?

15. A farmer wishes to put 364 bushels of corn and 455 bushels of oats into the least number of bins possible, that shall contain the same number of bushels without mixing the two kinds of grain; what number of bushels must each bin hold?

16. A man having a triangular piece of land, the sides of which are 165 feet, 231 feet, and 385 feet, wishes to inclose it with a fence having panels of the greatest possible uniform length; what will be the length of each panel?

17. B has $620, C $1116, and D $1488, with which they agree to purchase horses, at the highest price per head that will allow each man to invest all his money; how many horses can each man purchase?

18. How many rails will inclose a field 14599 feet long by 10361 feet wide, if the fence is straight, and 7 rails high, and the rails of equal length, and the longest that can be used?

19. A has $120, B $240, and C $384. They agree to purchase cows at the highest price per head that will allow each man to invest all his money. What price must they pay and how many cows can each purchase?

20. A village street is 332 rods long; A owns 124 rods front, B 116 rods, and C 92 rods; they agree to divide their land into equal lots of the largest size that will allow each one to form an exact number of lots. What will be the width of the lots?

21. A speculator has 3 fields, the first containing 18, the second 24, and the third 40 acres, which he wishes to divide into the largest possible lots having the same number of acres in each. How many acres will there be in each lot?

22. A farmer had 231 bushels of wheat and 273 bushels of oats, which he wished to put into the least number of bins containing the same number of bushels without mixing the two kinds. What number of bushels must each bin hold?

23. A street 399 ft. long and 35 ft. wide is to be paved with square flagstones of equal size, and as large as possible. How long and wide must each flagstone be?

24. A forwarding merchant has 2722 bushels of wheat, 1822 bushels of corn, and 1226 bushels of oats, which he wishes to forward in the fewest bags of equal size that will exactly hold any of the three kinds of grain. How many bags will it take?

25. A man having on deposit $182, $234, and $390 respectively in 3 different banks, wishes to draw out the whole in equal sums as large as possible. What is the greatest sum for which he must draw his checks?

LEAST COMMON MULTIPLE.

223. A **Multiple** is a number exactly divisible by a given number; thus, 20 is a multiple of 4.

NOTE.—A multiple is necessarily composite; a divisor may be either prime or composite. A number is a divisor of all its multiples and a multiple of all its divisors.

224. A **Common Multiple** is a number exactly divisible by two or more given numbers.

NOTE.—A common multiple of numbers may be found by finding their product.

225. The **Least Common Multiple** of two or more numbers is the least number exactly divisible by those numbers; thus, 24 is the least common multiple of 3, 4, 6, and 8.

Examples.

226. To find the least common multiple.—First method.

I. *A multiple of a number must contain all its prime factors.*

II. *A common multiple of two or more numbers must contain all the prime factors of each of those numbers.*

III. *The least common multiple of two or more numbers must contain all the prime factors of each of those numbers, and no other factors.*

1. Find the least common multiple of 63, 66, and 78.

OPERATION.

$$78 = 2 \times 3 \times 13$$
$$66 = 2 \times 3 \times 11$$
$$63 = 3 \times 3 \times 7$$
$$2 \times 3 \times 13 \times 11 \times 3 \times 7 = 18018 \textit{ Ans.}$$

SOLUTION.—The number cannot be less than 78, because it must contain 78; and if it contains 78, it must contain all its prime factors, viz.: $2 \times 3 \times 13$. We here have all the prime factors, and also all factors of 66 except 11. Annexing 11 to the series of factors we have $2 \times 3 \times 13 \times 11$, all the prime factors of 78 and 66, and all the factors of 63 except one 3, and 7. Annexing 3 and 7 to the series we have $2 \times 3 \times 13 \times 11 \times 3 \times 7$, all the prime factors of each of the given numbers, and *no others*. The product of this series of factors is the least common multiple of the numbers (III).

RULE.—I. *Resolve the given numbers into their prime factors.*

II. *Find the product of all the prime factors of the largest number, and such prime factors of the other numbers as are not found in the largest number.*

NOTE. — When a prime factor is repeated in any number, it must be taken as many times in the multiple, as the greatest number of times it appears in any of the given numbers.

227. To find the least common multiple. — Second method.

1. What is the least common multiple of 4, 9, 12, 18, and 36?

OPERATION.

2	4	9	12	18	36
2	2	9	6	9	18
3		9	3	9	9
3		3		3	[illegible]

$2 \times 2 \times 3 \times 3 = 36$ *Ans.*

SOLUTION. — We first write the given numbers in a series with a vertical line at the left. Since 2 is a factor of *some* of the given numbers, it must be a factor of the least common multiple sought (§ 226, III). Dividing as many of the numbers as are divisible by 2, we write the quotients, and the undivided number, 9, in a line underneath. Now, since some of the numbers in the second line contain the factor 2, the least common multiple must contain another 2, and we again divide by 2, omitting to write any quotient when it is 1. We next divide by 3 for a like reason, and again by 3. By this process we have transferred all the factors of each of the numbers to the left of the vertical line; and their product, 36, must be the least common multiple.

2. What is the least common multiple of 20, 12, 15, and 75?

OPERATION.

2, 5	20	12	15	75
2, 3	2	6	3	15
5				5

$2 \times 5 \times 2 \times 3 \times 5 = 300$ *Ans.*

SOLUTION. — We readily see that 2 and 5 are among the factors of the given numbers, and must be factors of the least common multiple; hence, writing 2 and 5 at the left, we divide every number that is divisible by *either* of these factors or by their *product*; thus, we divide 20 by both 2 and 5; 12 by 2; 15 by 5; and 75 by 5. We next divide the second line in like manner by 2 and 3; and afterward the third line by 5. By this process we collect the factors of the given numbers into *groups*; and the product of the factors at the left of the vertical is the least common multiple sought.

3. What is the least common multiple of 7, 10, 15, 42, and 70?

OPERATION.

3, 7	15	42	70
2, 5	5	2	10

$3 \times 7 \times 2 \times 5 = 210$ *Ans.*

SOLUTION. — In this operation we omit the 7 and 10, because they are exactly contained in some of the other given numbers; thus, 7 is contained in 42, and 10 in 70; and whatever will contain 42 and 70 must contain 7 and 10. Hence we have only to find the least common multiple of the remaining numbers, 15, 42, and 70, in the same way as in Ex. 2, and we find the answer to be 210.

Rule. — I. *Write the numbers in a line, omitting any of the smaller numbers that are factors of the larger, and draw a vertical line at the left.*

II. *Divide by any prime factor or factors that may be contained in one or more of the given numbers, and write the quotients and undivided numbers in a line underneath, omitting the 1's.*

III. *In like manner divide the quotients and undivided numbers, and continue the process till all the factors of the given numbers have been transferred to the left of the vertical line. Then find the product of these factors, which will be the least common multiple required.*

Note. — We may use a composite number for a divisor, when it is contained in all the given numbers.

What is the least common multiple of:

4. 18, 27, 36, and 40?
5. 12, 26, and 52?
6. 32, 34, and 36?
7. 8, 12, 18, 24, 27, and 36?
8. 22, 33, 44, 55, and 66?
9. 64, 84, 96, and 216?
10. 15, 18, 21, 24, 35, 36, 42, 50, and 60?
11. 6, 8, 10, 15, 18, 20, 24 and 30?

12. If A can build 14 rods of fence in a day, B 25 rods, C 8 rods, and D 20 rods, what is the least number of rods that will furnish a number of whole days' work to either one of the four men?

13. A can dig 4 rods of ditch in a day, B can dig 8 rods, and C can dig 6 rods. What must be the length of the shortest ditch that will furnish exact days' labor either for each working alone or for all working together?

14. The forward wheel of a carriage was 11 feet in circumference, and the hind wheel 15 feet; a rivet in the tire of each was up when the carriage started, and when it stopped the same rivets were up together for the 575th time. How many miles had the carriage traveled, allowing 5280 feet to the mile?

CANCELLATION.

228. **Cancellation** is the process of rejecting equal factors from numbers sustaining to each other the relation of dividend and divisor (§169, III).

Examples.

229. To divide by cancellation.

1. Divide 1365 by 105.

OPERATION.

$$\frac{1365}{105}=\frac{\cancel{3}\times\cancel{5}\times\cancel{7}\times 13}{\cancel{3}\times\cancel{5}\times\cancel{7}}=13 \quad \textit{Ans.}$$

Solution. — We first indicate the division by writing the dividend above a horizontal line and the divisor below. Then factoring each term, we find that 3, 5, and 7 are common factors; and crossing, or canceling these factors, we have 13, the remaining factor of the dividend, for a quotient.

Note. — If the product of several numbers is to be divided by the product of several other numbers, the common factors should be canceled before the multiplications are performed. The operations in multiplication and division will thus be abridged, and the factors of small numbers are generally more readily detected than those of large numbers.

2. Divide 20 times 56 by 7 times 15.

OPERATION.

$$\frac{\overset{4}{\cancel{20}}\times\overset{8}{\cancel{56}}}{\cancel{7}\times\underset{3}{\cancel{15}}}=\frac{32}{3}=10\tfrac{2}{3} \quad \textit{Ans.}$$

Solution. — Having first indicated all the operations required by the question, we cancel 7 from 7 and 56, and 5 from 15 and 20, leaving the factor 3 in the divisor, and 8 and 4 in the dividend. Then $8\times 4=32$, which, divided by 3, gives $10\frac{2}{3}$, the quotient required.

Note. — When a factor is canceled, the unit, 1, is supposed to take its place. It is thought more convenient by some to write the factors of the dividend on the right of a *vertical* line, and the factors of the divisor on the left.

3. What is the quotient of $18\times 6\times 4\times 42$ divided by $4\times 9\times 3\times 7\times 6$?

FIRST OPERATION.

$$\frac{\overset{2}{\cancel{18}}\times\cancel{6}\times\cancel{4}\times\overset{2}{\cancel{42}}}{\cancel{4}\times\cancel{9}\times\cancel{3}\times\cancel{7}\times\cancel{6}}=4 \quad \textit{Ans.}$$

SECOND OPERATION.

$$\begin{array}{r|l}
\cancel{4} & \cancel{18}^{2} \\
\cancel{9} & \cancel{6} \\
\cancel{3} & \cancel{4} \\
\cancel{7} & \cancel{42}^{2} \\
\cancel{6} & \\
\hline
 & 4 \ \textit{Ans.}
\end{array}$$

RULE. — I. *Write the numbers composing the dividend on one side of a horizontal or vertical line, and the numbers composing the divisor on the other.*

II. *Cancel all the factors common to both dividend and divisor, and then divide the product of the remaining factors of the dividend by the product of the remaining factors of the divisor.*

4. Divide the product of $21 \times 8 \times 60 \times 8 \times 6$ by $7 \times 12 \times 3 \times 8 \times 3$.

5. The product of the numbers 16, 5, 14, 40, 16, 60, and 50 is to be divided by the product of the numbers 40, 24, 50, 20, 7, and 10; what is the quotient?

6. Divide the continued product of 12, 5, 183, 18, and 70 by the continued product of 3, 14, 9, 5, 20, and 6.

7. If $213 \times 84 \times 190 \times 264$ is divided by $30 \times 56 \times 36$, what will be the quotient?

8. Multiply 64 times 7 by 31 and divide the product by 8 times 56; multiply this quotient by 15 times 88 and divide the product by 55; multiply this quotient by 13 and divide the product by 4 times 6.

9. How many firkins of butter, each containing 56 pounds at 15 cents a pound, must be given for 8 barrels of sugar, each containing 195 pounds, at 4 cents a pound?

10. A merchant sold to a farmer two kinds of cloth, one kind at 75 cents a yard, and the other at 90 cents, selling him twice as many yards of the first kind as of the second. He received as pay 132 pounds of butter at 20 cents a pound; how many yards of each kind of cloth did he sell?

11. A man took six loads of potatoes to market, each load containing 20 bags, and each bag 2 bushels. He sold them at 44 cents a bushel, and received in payment 8 chests of tea, each containing 22 pounds; how much was the tea worth a pound?

12. A grocer sold 25 boxes of soap, each containing 66 pounds, at 9 cents a pound, and received as pay 99 barrels of potatoes, each containing 3 bushels; how much were the potatoes worth a bushel?

FRACTIONS.

DEFINITIONS, NOTATION, AND NUMERATION.

230. When it is necessary to express a quantity less than a unit, we may regard the unit as divided into some number of equal parts, and use one of these parts as a new unit of less value than the unit divided. Thus, if a yard, considered as an integral unit, is divided into 4 equal parts, then one, two, or three of these parts will constitute a number less than a unit. The *parts of a unit* thus used are called *fractional units;* and the numbers formed from them, *fractional numbers.*

231. A **Fractional Unit** is *one* of the equal parts of an integral unit.

232. A **Fraction** is a fractional unit, or a collection of fractional units.

Fractional units take their *name*, and their *value*, from the *number* of parts into which the integral unit is divided.

If a unit is divided into 2 equal parts, one of the parts is called *one half.* If a unit is divided into 3 equal parts, one of the parts is called *one third.* If a unit is divided into 4 equal parts, one of the parts is called *one fourth.* And it is evident that one *third* is less in value than one *half*, one *fourth* less than one *third*, and so on.

233. To express a fraction by figures, two integers are required; one to denote the number of parts into which the integral unit is divided, the other to denote the number of parts taken, or the number of fractional units in the collection. The former is written below a horizontal line, the latter above.

One half is written		$\frac{1}{2}$	One sixth is written		$\frac{1}{6}$
One third	"	$\frac{1}{3}$	Five sixths	"	$\frac{5}{6}$
Two thirds	"	$\frac{2}{3}$	One seventh	"	$\frac{1}{7}$
One fourth	"	$\frac{1}{4}$	Three sevenths	"	$\frac{3}{7}$
Two fourths	"	$\frac{2}{4}$	Three eighths	"	$\frac{3}{8}$
Three fourths	"	$\frac{3}{4}$	Five ninths	"	$\frac{5}{9}$
One fifth	"	$\frac{1}{5}$	Eight tenths	"	$\frac{8}{10}$
Two fifths	"	$\frac{2}{5}$	One twelfth	"	$\frac{1}{12}$

234. The **Denominator** of a fraction is the number below the line. It denominates or names the fractional unit, and it shows how many fractional units are equal to an integral unit.

A **Common Denominator** is a denominator common to two or more fractions, and the **Least Common Denominator** is the least denominator common to them.

235. The **Numerator** is the number above the line. It numerates or numbers the fractional units; and it shows how many are taken.

236. The **Terms** of a fraction are the numerator and denominator, taken together.

237. **Similar Fractions** are fractions that have the same denominator, as $\frac{3}{8}$ and $\frac{5}{8}$. **Dissimilar Fractions** have different denominators, as $\frac{1}{3}$ and $\frac{1}{4}$.

238. A fraction is in its **Lowest Terms** when its numerator and denominator are prime to each other, that is, when the terms have no common divisor.

239. The denominator of a fraction shows how many fractional units in the numerator are equal to 1 integral unit, hence we have the following principles:

PRINCIPLES.—I. *The value of a fraction in integral units is equal to the quotient of the numerator divided by the denominator.*

II. *Fractions indicate division, the numerator being a dividend, the denominator a divisor, and the value of a fraction, a quotient.*

240. To analyze a fraction is to designate and describe its numerator and denominator. Thus $\frac{5}{7}$ is analyzed as follows:

7 is the *denominator*, and shows that the units expressed by the numerator are *sevenths*.

5 is the *numerator*, and shows that 5 sevenths are taken.

5 and 7 are the *terms* of the fraction considered as an expression of division, 5 being the dividend and 7 the divisor.

241. A **Proper Fraction** is one whose numerator is less than its denominator; its value is less than the unit 1; as, $\frac{7}{8}$, $\frac{5}{12}$.

242. An **Improper Fraction** is one whose numerator equals or exceeds its denominator; its value is never less than the unit 1; as, $\frac{8}{8}$, $\frac{9}{8}$.

NOTES. — 1. The value of a proper fraction, always being less than a unit, can only be expressed in a fractional form; hence, its name.

2. The value of an improper fraction, always being equal to, or greater than a unit, can always be expressed in some other form; hence, its name.

243. A **Mixed Number** is a number expressed by an integer and a fraction; as, $5\frac{1}{2}$, $16\frac{2}{7}$.

Examples.

244. Express the following fractions by figures:

1. Four *ninths.*
2. Seven *fifty-sixths.*
3. Sixteen *forty-eighths.*
4. Ninety-five *one hundred seventy-ninths.*
5. Five hundred thirty-six *four hundredths.*
6. One thousand eight hundred fifty-seven *nine thousand five hundred twenty-firsts.*
7. Twenty-five thousand *eighty-sevenths.*
8. Thirty *ten thousand eighty-seconds.*
9. One hundred one *ten millionths.*
10. Two thousand fifty-six *thirty-three thousand five hundred twenty-fifths.*
11. Thirty thousand seven hundred fifty-nine *fifty thousand two hundred nineteenths.*
12. Six thousand seven hundred seventy-seven *nine thousand two hundred fifty-thirds.*

13. One hundred one thousand two hundred fifteen *three hundred three thousand twenty-fifths.*

14. Eight thousand three hundred fifty-four *twenty-seven thousand three hundred seventy-fourths.*

Read and analyze the following fractions:

15. $\frac{4}{9}$; $\frac{7}{12}$; $\frac{17}{38}$; $\frac{45}{100}$; $\frac{72}{375}$; $\frac{48}{1009}$; $\frac{84}{7863}$; $\frac{456}{587}$.

16. $\frac{20}{4}$; $\frac{87}{30}$; $\frac{95}{100}$; $\frac{48}{12}$; $\frac{75}{437}$; $\frac{175}{2}$; $\frac{436}{60}$; $\frac{766}{4879}$.

17. $\frac{467}{936}$; $\frac{536}{248}$; $\frac{10000}{75}$; $\frac{75}{10000}$; $\frac{5007}{8007}$.

18. $\frac{150}{587}$; $\frac{436}{972}$; $\frac{13785}{47956}$; $\frac{150072}{475000}$; $\frac{100001}{200002}$.

19. $\frac{231}{436}$; $\frac{312}{820}$; $\frac{14221}{56019}$; $\frac{22009}{100000}$; $\frac{1000000}{5000000}$.

20. $\frac{145}{350}$; $\frac{218}{927}$; $\frac{89765}{98756}$; $\frac{100005}{250009}$; $\frac{876543}{1820909}$.

245. Since fractions indicate division (§ 239, II), all changes in the *terms* of a fraction will affect the *value* of that fraction according to the laws of division; and we only need to modify the language of the General Principles of Division, by substituting the words *numerator, denominator,* and *fraction,* or *value of the fraction,* for the words *dividend, divisor,* and *quotient,* respectively (§ 169), and we shall have the following principles:

PRINCIPLES.—I. *Multiplying the numerator multiplies the fraction, and dividing the numerator divides the fraction.*

II. *Multiplying the denominator divides the fraction, and dividing the denominator multiplies the fraction.*

III. *Multiplying or dividing both terms of the fraction by the same number, does not alter the value of the fraction.*

246. These three principles may be embraced in one general law (§ 170).

GENERAL LAW.—*A change in the* NUMERATOR *produces a* LIKE *change in the value of the fraction; but a change in the* DENOMINATOR *produces an* OPPOSITE *change in the value of the fraction.*

REDUCTION.

247. **Reduction** of fractions is the process of changing their terms, or their forms, without altering their value.

Examples.

248. **To reduce fractions to their lowest terms.**

1. Reduce the fraction $\frac{60}{105}$ to its lowest terms.

OPERATION.

$\frac{60}{105}=\frac{12}{21}=\frac{4}{7}$ *Ans.*

Or,

$15)\frac{60}{105}=\frac{4}{7}$ *Ans.*

SOLUTION. — Dividing both terms of the fraction by the same number does not alter the value of the fraction (§ 245, III); hence, we divide both terms of $\frac{60}{105}$ by 5, and both terms of the result, $\frac{12}{21}$, by 3, and obtain $\frac{4}{7}$ for the final result. As 4 and 7 are *prime* to each other, the lowest terms of $\frac{60}{105}$ are $\frac{4}{7}$. Instead of dividing by the factors 5 and 3 successively, we may divide by the greatest common divisor of the terms 60 and 105, which is 15, and reduce the fraction to its lowest terms at a single operation.

RULE. — *Cancel all factors common to numerator and denominator; or, divide both terms by their greatest common divisor.*

Reduce to their lowest terms:

2. $\frac{72}{126}$. 4. $\frac{75}{135}$. 6. $\frac{252}{287}$. 8. $\frac{247}{468}$. 10. $\frac{4488}{5661}$.
3. $\frac{84}{96}$. 5. $\frac{156}{208}$. 7. $\frac{629}{1737}$. 9. $\frac{851}{1219}$. 11. $\frac{3277}{5617}$.

249. **To reduce an improper fraction to a whole or mixed number.**

1. Reduce $\frac{297}{12}$ to a whole or mixed number.

OPERATION.

$\frac{297}{12}=297 \div 12=24\frac{9}{12}=24\frac{3}{4}$. *Ans.*

SOLUTION. — Since the value of a fraction in integral units is equal to the quotient of the numerator divided by the denominator (§ 239, I), we divide the given numerator, 297, by the given denominator, 12, and obtain for the value of the fraction the mixed number $24\frac{9}{12}=24\frac{3}{4}$.

RULE. — *Divide the numerator by the denominator.*

NOTES. — 1. When the denominator is an exact divisor of the numerator, the result will be a whole number; in all other cases, a mixed number.

2. In all answers containing fractions, reduce the fractions to their lowest terms.

Change to equivalent integers or mixed numbers:

2. $\frac{48}{6}$. 4. $\frac{349}{12}$. 6. $\frac{2481}{65}$. 8. $\frac{3828}{566}$. 10. $\frac{7852}{8}$.
3. $\frac{48}{4}$. 5. $\frac{538}{24}$. 7. $\frac{10315}{164}$. 9. $\frac{22935}{226}$. 11. $\frac{483}{16}$.

250. To reduce a whole number to a fraction having a given denominator.

1. Reduce 37 to an equivalent fraction with a denominator of 5.

OPERATION.

$37 \times 5 = 185$

$37 = \frac{185}{5}$ *Ans.*

SOLUTION. — Since in each unit there are 5 fifths, in 37 units there must be 37 times 5 fifths, or 185 fifths $= \frac{185}{5}$. The numerator, 185, is obtained in the operation by multiplying the whole number, 37, by the given denominator.

RULE. — *Multiply the whole number by the given denominator; take the product for a numerator, under which write the given denominator.*

NOTE. — A whole number may be reduced to a fractional form by writing 1 under it for a denominator; thus, $9 = \frac{9}{1}$.

2. Reduce 17 to an equivalent fraction having 6 for its denominator.
3. Change 375 to a fraction having a denominator of 8.
4. Change 478 to a fraction having 24 for its denominator.
5. Reduce 36 pounds to ninths of a pound.
6. Reduce 359 days to sevenths of a day.
7. Reduce 763 feet to fourteenths of a foot.
8. Change 312 miles to sixteenths of a mile.

251. To reduce a mixed number to an improper fraction.

1. In $12\frac{5}{7}$ how many sevenths are there?

OPERATION.

$$\begin{array}{r} 12\frac{5}{7} \\ 7 \\ \hline \frac{89}{7} \end{array} \quad \textit{Ans.}$$

SOLUTION. — In the whole number 12, there are 12×7 sevenths = 84 sevenths (§ 250), and 84 sevenths + 5 sevenths = 89 sevenths, or $\frac{89}{7}$.

RULE. — *Multiply the whole number by the denominator of the fraction; to the product add the numerator, and under the sum write the denominator.*

Change to improper fractions:

2. $15\frac{4}{5}$.	5. $356\frac{13}{17}$.	8. $15\frac{7}{8}$.	11. $760\frac{9}{10}$.
3. $24\frac{3}{4}$.	6. $300\frac{1}{300}$.	9. $135\frac{9}{25}$.	12. $1208\frac{3}{64}$.
4. $57\frac{4}{7}$.	7. $434\frac{11}{18}$.	10. $872\frac{5}{12}$.	13. $2100\frac{11}{16}$.

252. To reduce a fraction to a given denominator.

We have seen that fractions may be reduced to lower terms by division. Conversely,

Principles. — I. *Fractions may be reduced to higher terms by multiplication.*

II. *All higher terms of a fraction must be multiples of its lowest terms.*

1. Reduce $\frac{3}{8}$ to a fraction whose denominator is 40.

OPERATION.

$40 \div 8 = 5$

$\frac{3 \times 5}{8 \times 5} = \frac{15}{40}$ *Ans.*

Solution. — We first divide 40, the required denominator, by 8, the denominator of the given fraction, to ascertain if it is a multiple of this term, 8. The division shows that it is a multiple, and that 5 is the factor which must be employed to produce it. We therefore multiply both terms of $\frac{3}{8}$, by 5 (§ 245, III), and obtain $\frac{15}{40}$, the required result.

Rule. — *Divide the required denominator by the denominator of the given fraction, and multiply both terms of the fraction by the quotient.*

2. Reduce $\frac{3}{8}$ to a fraction having 24 for a denominator.
3. Reduce $\frac{7}{12}$ to a fraction whose denominator is 96.
4. Reduce $\frac{12}{17}$ to a fraction whose denominator is 51.
5. Reduce $\frac{9}{13}$ to a fraction whose denominator is 78.
6. Reduce $\frac{62}{375}$ to a fraction whose denominator is 3000.
7. Change $7\frac{3}{4}$ to a fraction whose denominator is 8.
8. Change $16\frac{7}{28}$ to a fraction whose denominator is 176.

253. To reduce two or more fractions to a common denominator.

1. Reduce $\frac{3}{5}$ and $\frac{7}{9}$ to a common denominator.

OPERATION.

$\frac{3 \times 9}{5 \times 9} = \frac{27}{45}$

$\frac{7 \times 5}{9 \times 5} = \frac{35}{45}$

Solution. — We multiply both terms of the first fraction by the denominator of the second, and both terms of the second fraction by the denominator of the first (§ 245, III). This must reduce each fraction to the same denominator, for each new denominator will be the product of the given denominators.

Rule. — *Multiply the terms of each fraction by the denominators of all other fractions.*

Note. — Mixed numbers must first be reduced to improper fractions.

Reduce to fractions having a common denominator:

2. $\frac{2}{3}$, $\frac{2}{5}$. 4. $\frac{3}{5}$, $\frac{5}{12}$, $\frac{1}{2}$. 6. $\frac{1}{2}$, $5\frac{2}{3}$, $1\frac{3}{5}$.
3. $\frac{3}{7}$, $\frac{5}{9}$. 5. $\frac{3}{7}$, $\frac{2}{3}$, $\frac{1}{4}$. 7. $\frac{1}{9}$, $1\frac{1}{4}$, $8\frac{1}{3}$.

254. To reduce fractions to their least common denominator.

PRINCIPLES.—I. *If two or more fractions are reduced to a common denominator, this common denominator will be a common multiple of the several denominators.*

II. *The least common denominator must therefore be the least common multiple of the several denominators.*

1. Reduce $\frac{5}{6}$, $\frac{7}{12}$, and $\frac{2}{15}$ to their least common denominator.

OPERATION.

$$\begin{array}{r|ll} 3,\ 5 & 12 & 15 \\ \hline 2,\ 2 & 4 & \\ \hline \end{array}$$

$$3 \times 5 \times 2 \times 2 = 60$$

$$\left.\begin{array}{r} \frac{5}{6} = \frac{50}{60} \\ \frac{7}{12} = \frac{35}{60} \\ \frac{2}{15} = \frac{8}{60} \end{array}\right\} \textit{Ans.}$$

SOLUTION.—We first find the least common multiple of the given denominators (§ 227), which is 60. This must be the least common denominator to which the given fractions can be reduced (II). Reducing each fraction to the denominator 60, by § 252, we obtain $\frac{50}{60}$, $\frac{35}{60}$, and $\frac{8}{60}$ for the answer.

RULE.—I. *Find the least common multiple of the given denominators, for the least common denominator.*

II. *Divide this common denominator by each of the given denominators, and multiply each numerator by the corresponding quotient. The products will be the new numerators.*

NOTES.—1. If the several fractions are not in their lowest terms, they should be reduced to their lowest terms before applying the rule.

2. When two or more fractions are reduced to their least common denominator, their numerators and the common denominator will be prime to each other.

Reduce to their least common denominator:

2. $\frac{5}{6}$, $\frac{3}{10}$. 6. $\frac{2}{3}$, $\frac{4}{13}$, $\frac{25}{26}$, $\frac{4}{39}$. 10. $\frac{161}{529}$, $\frac{289}{391}$, $\frac{1147}{1961}$.
3. $\frac{2}{3}$, $\frac{3}{4}$, $\frac{5}{6}$. 7. $2\frac{3}{5}$, $\frac{7}{15}$, $\frac{5}{24}$, $\frac{37}{60}$. 11. $2\frac{5}{7}$, $\frac{3}{14}$, $1\frac{7}{10}$.
4. $\frac{3}{5}$, $\frac{7}{12}$, $\frac{11}{15}$. 8. $\frac{20}{21}$, $\frac{9}{56}$, $\frac{51}{84}$, $\frac{5}{42}$. 12. $\frac{931}{1829}$, $\frac{8127}{5723}$, $\frac{5133}{5626}$.
5. $\frac{2}{3}$, $\frac{8}{9}$, $\frac{3}{8}$. 9. $\frac{25}{40}$, $\frac{25}{120}$, $\frac{14}{64}$, $\frac{1}{8}$. 13. $\frac{5}{7}$, $\frac{11}{12}$, $\frac{2}{15}$, $\frac{8}{27}$, $\frac{9}{35}$, $\frac{17}{40}$.

ADDITION.

255. The denominator of a fraction determines the value of the fractional unit (§ 234); hence,

PRINCIPLES. — I. *If two or more fractions have the same denominator, their numerators express fractional units of the same value.*

II. *If two or more fractions have different denominators, their numerators express fractional units of different values.*

III. *Fractions can be added only when they have a common denominator.*

Examples.

256. To add fractions or mixed numbers.

1. What is the sum of $\frac{1}{24}$, $\frac{5}{24}$, and $\frac{7}{24}$?

OPERATION.

$$\frac{1}{24}+\frac{5}{24}+\frac{7}{24}=\frac{1+5+7}{24}=\frac{13}{24} \quad \textit{Ans.}$$

SOLUTION. — Since the denominators are alike, we simply add the numerators, and place the result over the denominator.

2. What is the sum of $\frac{1}{5}$, $\frac{5}{12}$, and $\frac{2}{15}$?

OPERATION.

$$\frac{1}{5}+\frac{5}{12}+\frac{2}{15}=\frac{12+25+8}{60}=\frac{45}{60}=\frac{3}{4}.$$

SOLUTION. — We first reduce the given fractions to their least common denominator (§ 254). As the resulting fractions, $\frac{12}{60}$, $\frac{25}{60}$, and $\frac{8}{60}$ have the same fractional unit, we add them by uniting their numerators into one sum, making $\frac{45}{60}=\frac{3}{4}$, the answer.

3. Add $5\frac{3}{4}$, $3\frac{7}{8}$, $4\frac{7}{12}$.

OPERATION.

$$\begin{array}{l} 5+3+4=12 \\ \frac{3}{4}+\frac{7}{8}+\frac{7}{12}=\underline{2\frac{5}{24}} \\ \qquad\qquad 14\frac{5}{24} \quad \textit{Ans.} \end{array}$$

SOLUTION. — The sum of the integers, 5, 3, and 4, is 12; the sum of the fractions, $\frac{3}{4}$, $\frac{7}{8}$, and $\frac{7}{12}$, is $2\frac{5}{24}$. Hence, the sum of both fractions and integers is $12+2\frac{5}{24}=14\frac{5}{24}$.

NOTE. — All fractional results should be reduced to their lowest terms, and all improper fractions to whole or mixed numbers.

RULE. — I. To add fractions. — *Reduce the fractions to their least common denominator; then add the numerators, and place the sum over the common denominator.*

II. To add mixed numbers. — *Add the integers and fractions separately, and then add their sums.*

Add:

4. $\frac{7}{12}$, $\frac{4}{12}$, $\frac{5}{12}$, $\frac{11}{12}$.
5. $\frac{13}{15}$, $\frac{4}{15}$, $\frac{2}{15}$, $\frac{8}{15}$.
6. $\frac{5}{21}$, $\frac{8}{21}$, $\frac{16}{21}$, $\frac{19}{21}$.
7. $7\frac{11}{36}$, $8\frac{23}{36}$, $2\frac{17}{36}$, $5\frac{19}{36}$.
8. $37\frac{9}{56}$, $12\frac{27}{56}$, $13\frac{37}{56}$, $\frac{53}{56}$.
9. $9\frac{1}{40}$, $\frac{18}{40}$, $6\frac{3}{40}$, $7\frac{29}{40}$.
10. $18\frac{1}{72}$, $7\frac{9}{72}$, $3\frac{15}{72}$, 14.
11. $\frac{2}{3}$, $\frac{3}{4}$, $\frac{5}{6}$, $\frac{7}{8}$, $\frac{1}{2}$.
12. $\frac{1}{3}$, $\frac{2}{9}$, $\frac{1}{15}$, $\frac{1}{5}$, $\frac{2}{3}$.
13. $\frac{3}{5}$, $\frac{4}{7}$, $\frac{18}{21}$, $\frac{1}{3}$.
14. $\frac{5}{12}$, $\frac{13}{15}$, $\frac{7}{20}$.
15. $\frac{1}{3}$, $\frac{3}{4}$, $\frac{11}{12}$, $\frac{29}{42}$.
16. $3\frac{1}{2}$, $4\frac{2}{3}$, $2\frac{2}{15}$.
17. $1\frac{1}{2}$, $2\frac{2}{3}$, $3\frac{3}{4}$, $4\frac{4}{5}$.
18. $4\frac{7}{15}$, $8\frac{5}{21}$, $2\frac{8}{35}$.
19. $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{12}$, $\frac{1}{17}$.
20. $\frac{1}{2}$, $\frac{2}{3}$, $\frac{2}{13}$, $\frac{5}{17}$.
21. $\frac{1}{2}$, $\frac{2}{9}$, $\frac{3}{11}$, $\frac{7}{18}$.
22. $\frac{1}{2}$, $\frac{5}{14}$, $\frac{8}{19}$, $\frac{2}{3}$, $\frac{3}{7}$, $\frac{1}{6}$, $\frac{3}{5}$.
23. $41\frac{1}{2}$, $105\frac{2}{9}$, $300\frac{3}{4}$, $241\frac{3}{5}$, $472\frac{1}{4}$.
24. $4\frac{1}{3}$, $2\frac{1}{4}$, $1\frac{1}{16}$, $2\frac{5}{24}$, $5\frac{7}{16}$, $7\frac{2}{3}$, $4\frac{1}{2}$, $6\frac{5}{6}$.
25. What is the sum of $13\frac{4}{7}$, $3\frac{1}{4}$, $8\frac{2}{3}$, and $5\frac{1}{5}$?
26. Find the sum of $\frac{140}{320}$, $\frac{57}{80}$, $1\frac{1}{10}$, $\frac{19}{20}$, and $\frac{105}{160}$.
27. Four cheeses weighed respectively $36\frac{5}{8}$, $42\frac{2}{3}$, $39\frac{7}{16}$, and $51\frac{1}{4}$ pounds. What was their entire weight?
28. What number is that from which if $4\frac{4}{5}$ are taken away the remainder will be $3\frac{23}{35}$?
29. What fraction is that which exceeds $\frac{5}{18}$ by $\frac{5}{54}$?
30. A farmer divides his farm into 5 fields; the first contains $26\frac{7}{12}$ acres, the second $40\frac{16}{21}$ acres, the third $51\frac{6}{7}$ acres, the fourth $59\frac{3}{4}$ acres, and the fifth $62\frac{3}{8}$ acres. How many acres are there in the farm?
31. A merchant sold $46\frac{4}{9}$ yards of cloth for \$ $127\frac{7}{16}$, $64\frac{14}{27}$ yards for \$ $226\frac{5}{8}$, and $76\frac{5}{6}$ yards for \$ $312\frac{3}{8}$. How many yards of cloth did he sell, and how much did he receive for the whole?

SUBTRACTION.

257. The process of subtracting one fraction from another is based upon the following principles:

PRINCIPLES. — I. *One number can be subtracted from another only when the two numbers have the same unit value.*

II. *In subtraction of fractions, the minuend and subtrahend must have a common denominator* (§ 255, I).

Examples.

258. To subtract fractions or mixed numbers.

1. From $\frac{7}{12}$ take $\frac{5}{12}$.

OPERATION.

$$\frac{7}{12}-\frac{5}{12}=\frac{7-5}{12}=\frac{2}{12}=\frac{1}{6}\ \textit{Ans.}$$

SOLUTION. — Since the fractions have a common denominator, we subtract the numerator of the smaller from that of the greater, and reduce the result to its lowest terms.

2. From $\frac{4}{5}$ subtract $\frac{2}{3}$.

OPERATION.

$$\frac{4}{5}-\frac{2}{3}=\frac{12-10}{15}=\frac{2}{15}\ \textit{Ans.}$$

SOLUTION. — Reducing the given fractions to a common denominator, the resulting fractions $\frac{12}{15}$ and $\frac{10}{15}$ express fractional units of the same value (§ 255, I). Then 12 fifteenths less 10 fifteenths = 2 fifteenths = $\frac{2}{15}$, the answer.

3. From $238\frac{1}{4}$ take $24\frac{5}{6}$.

OPERATION.

$$\begin{array}{rl} 238\frac{1}{4} = & 238\frac{3}{12} \\ 24\frac{5}{6} = & 24\frac{10}{12} \\ \hline & 213\frac{5}{12}\ \textit{Ans.} \end{array}$$

SOLUTION. — We first reduce the fractional parts, $\frac{1}{4}$ and $\frac{5}{6}$, to the common denominator, 12. Since we cannot take $\frac{10}{12}$ from $\frac{3}{12}$, we add $1 = \frac{12}{12}$, to $\frac{3}{12}$, making $\frac{15}{12}$. Then, $\frac{10}{12}$ subtracted from $\frac{15}{12}$ leaves $\frac{5}{12}$; and 24 from 237 leaves 213, hence, we have $213\frac{5}{12}$ for the entire remainder.

RULE. — I. To subtract fractions. — *Reduce the fractions to their least common denominator. Subtract the numerator of the subtrahend from the numerator of the minuend, and place the difference of the new numerators over the common denominator.*

II. To subtract mixed numbers. — *Reduce the fractional parts to a common denominator, and then subtract the fractional and integral parts separately.*

NOTE. — We may reduce mixed numbers to improper fractions, and subtract by the rule for fractions. But this method generally imposes the useless labor of reducing integral numbers to fractions, and fractions to integers again.

Find the remainders:

4. $\frac{7}{15} - \frac{2}{15}$.	11. $\frac{14}{39} - \frac{19}{65}$.	18. $75 - 4\frac{3}{7}$.
5. $\frac{35}{56} - \frac{21}{56}$.	12. $\frac{5}{21} - \frac{1}{15}$.	19. $82 - 7\frac{4}{9}$.
6. $\frac{35}{52} - \frac{9}{52}$.	13. $\frac{7}{12} - \frac{5}{42}$.	20. $18\frac{2}{9} - 5\frac{5}{6}$.
7. $\frac{5}{6} - \frac{3}{8}$.	14. $\frac{7}{64} - \frac{13}{120}$.	21. $26\frac{7}{24} - 25\frac{13}{15}$.
8. $\frac{3}{4} - \frac{7}{20}$.	15. $\frac{9}{35} - \frac{7}{80}$.	22. $28\frac{16}{63} - 3\frac{9}{14}$.
9. $\frac{13}{15} - \frac{7}{24}$.	16. $16\frac{5}{6} - 7\frac{1}{6}$.	23. $78\frac{7}{15} - 32\frac{2}{15}$.
10. $\frac{9}{14} - \frac{16}{63}$.	17. $36\frac{11}{24} - 8\frac{19}{24}$.	24. $89\frac{2}{13} - 12\frac{1}{7}$.

25. From a cask of oil containing $31\frac{1}{2}$ gallons, $17\frac{5}{8}$ gallons were drawn. How many gallons remained?

26. A farmer, having $450\frac{7}{10}$ acres of land, sold $304\frac{3}{4}$ acres. How many acres had he left?

27. If flour is bought for \$ $6\frac{1}{4}$ per barrel, and sold for \$ $7\frac{3}{8}$, what will be the gain per barrel?

28. From the sum of $\frac{5}{7}$ and $3\frac{1}{2}$, take the difference between $4\frac{1}{3}$ and $5\frac{1}{4}$.

29. The sum of two numbers is $26\frac{1}{4}$, and the less is $7\frac{1}{15}$. What is the greater?

30. What number is that to which if you add $18\frac{4}{7}$, the sum will be $97\frac{5}{9}$?

31. What number must you add to the sum of $126\frac{1}{7}$ and $240\frac{3}{4}$, to make $560\frac{5}{8}$?

32. What number is that which, added to the sum of $\frac{1}{8}$, $\frac{1}{12}$, and $\frac{1}{18}$, will make $\frac{25}{36}$?

33. To what fraction must $\frac{2}{5}$ be added, so that the sum may be $\frac{3}{5}$?

34. From a barrel of vinegar containing $31\frac{1}{2}$ gallons, $14\frac{7}{8}$ gallons were drawn. How much was then left?

35. A dealer bought a quantity of coal for \$ $140\frac{3}{8}$, and of lumber for \$ $456\frac{2}{3}$. He sold the coal for \$ $775\frac{1}{8}$, and the lumber for \$ $516\frac{3}{16}$. How much was his whole gain?

THEORY OF MULTIPLICATION AND DIVISION.

259. In multiplication and division of fractions, the various operations may be considered in two classes:

1. Multiplying or dividing a fraction.
2. Multiplying or dividing *by* a fraction.

260. The methods of multiplying and dividing fractions may be derived directly from the General Principles of Fractions (§ 245); as follows:

RULE. — I. To multiply a fraction. — *Multiply its numerator or divide its denominator* (§ 245, I and II).

II. To divide a fraction. — *Divide its numerator or multiply its denominator* (§ 245, I and II).

III. *Perform the required operation upon the numerator, or the opposite upon the denominator* (§ 246).

261. The methods of multiplying and dividing *by* a fraction may be deduced, as follows:

1. The value of a fraction is the quotient of the numerator divided by the denominator (§ 239, I). Hence,
2. The numerator alone is as many times the value of the fraction, as there are units in the denominator.
3. If, therefore, in *multiplying* by a fraction, we multiply by the numerator, this result will be *too great*, and must be divided by the denominator.
4. But if in *dividing* by a fraction, we divide by the numerator, the resulting quotient will be *too small*, and must be multiplied by the denominator.

Hence, the methods of multiplying and dividing *by* a fraction may be stated as follows:

RULE. — I. To multiply by a fraction. — *Multiply by the numerator and divide by the denominator.*

II. To divide by a fraction. — *Divide by the numerator and multiply by the denominator.*

III. *Perform the required operation by the numerator and the opposite by the denominator.*

MULTIPLICATION.

Examples.

262. To multiply fractions by integers, integers by fractions, or fractions by fractions.

1. Multiply $\frac{5}{12}$ by 4.

FIRST OPERATION.

$\frac{5}{12} \times 4 = \frac{20}{12} = 1\frac{2}{3}$ *Ans.*

SECOND OPERATION.

$\frac{5}{12} \times 4 = \frac{5}{3} = 1\frac{2}{3}$ *Ans.*

THIRD OPERATION.

$$\frac{5}{\underset{3}{\cancel{12}}} \times \frac{\cancel{4}}{1} = \frac{5}{3} = 1\tfrac{2}{3}\ \textit{Ans.}$$

SOLUTION.—In the first operation, we multiply the fraction by 4 by multiplying its numerator by 4; and in the second operation, we multiply the fraction by 4 by dividing its denominator by 4 (§ 260, I or III).

In the third operation, we express the multiplier in the form of a fraction, indicate the multiplication, and obtain the result by cancellation.

2. Multiply 21 by $\frac{4}{7}$.

FIRST OPERATION.

$21 \times \frac{4}{7} = \frac{84}{7} = 12$ *Ans.*

SECOND OPERATION.

$21 \times \frac{4}{7} = 3 \times 4 = 12$ *Ans.*

THIRD OPERATION.

$$\frac{\overset{3}{\cancel{21}}}{1} \times \frac{4}{\cancel{7}} = 12\ \textit{Ans.}$$

SOLUTION.—To multiply by $\frac{4}{7}$, we must multiply by 4 and divide by 7 (§ 261, I or III).

In the first operation, we first multiply 21 by 4, and then divide the product, 84, by 7.

In the second operation, we first divide 21 by 7, and then multiply the quotient, 3, by 4.

In the third operation, we express the whole number, 21, in the form of a fraction, indicate the multiplication, and obtain the result by cancellation.

3. Multiply $\frac{5}{14}$ by $\frac{7}{8}$.

FIRST OPERATION.

1st step, $\frac{5}{14} \times 7 = \frac{35}{14}$

2d step, $\frac{35}{14} \div 8 = \frac{35}{112}$

$\frac{35}{112} = \frac{5}{16}$ *Ans.*

SECOND OPERATION.

$\frac{5}{14} \times \frac{7}{8} = \frac{35}{112} = \frac{5}{16}$ *Ans.*

THIRD OPERATION.

$$\frac{5}{\underset{2}{\cancel{14}}} \times \frac{\cancel{7}}{8} = \frac{5}{16}\ \textit{Ans.}$$

SOLUTION.—To multiply by $\frac{7}{8}$, we must multiply by 7 and divide by 8 (§ 261, I or III). In the first operation, we multiply $\frac{5}{14}$ by 7 and obtain $\frac{35}{14}$; we then divide $\frac{35}{14}$ by 8 and obtain $\frac{35}{112}$, which, reduced, gives $\frac{5}{16}$, the required product.

In the second operation, we obtain the same result by finding the product of the numerators and the product of the denominators.

In the third operation, we *indicate* the multiplication, and obtain the result by cancellation.

RULE.—I. *Reduce all integers and mixed numbers to improper fractions.*

II. *Find the product of the numerators for a new numerator, and of the denominators for a new denominator.*

NOTES.—1. Cancel all factors common to numerators and denominators.
2. If a fraction is multiplied by its denominator, the product will be the numerator.

Find the products in their lowest terms:

4. $\frac{3}{5}\times 8$.
5. $\frac{4}{9}\times 27$.
6. $\frac{3}{16}\times 4$.
7. $\frac{5}{36}\times 9$.
8. $\frac{2}{75}\times 15$.
9. $8\times\frac{3}{4}$.
10. $75\times\frac{3}{15}$.
11. $7\times\frac{8}{21}$.
12. $756\times\frac{5}{6}$.
13. $572\times\frac{5}{24}$.
14. $\frac{3}{7}\times\frac{5}{6}$.
15. $\frac{11}{15}\times\frac{20}{33}$.
16. $2\frac{4}{7}\times 3\frac{3}{5}\times 1\frac{1}{2}$.
17. $\frac{9}{17}\times 2\frac{13}{26}\times\frac{2}{3}$.
18. $\frac{2}{3}\times\frac{5}{6}\times\frac{9}{15}\times\frac{7}{8}$.
19. $\frac{1}{2}\times\frac{4}{5}\times\frac{3}{7}\times\frac{4}{5}$.
20. $\frac{2}{3}\times\frac{7}{8}\times\frac{15}{28}\times\frac{4}{11}\times\frac{44}{75}$.
21. $\frac{24}{35}\times\frac{7}{58}\times\frac{108}{119}$.
22. $2\frac{2}{5}\times 2\frac{4}{7}\times\frac{2}{11}\times\frac{5}{108}$.
23. $\frac{7}{11}\times\frac{5}{21}\times 4\frac{2}{5}\times 15$.
24. $\frac{43}{126}\times\frac{7}{860}\times\frac{540}{53}$.
25. Find the value of $\overline{(4\frac{1}{2}\times\frac{7}{8})+1\frac{3}{5}}\times(3\frac{1}{3}-\frac{9}{10})$.
26. Find the value of $28+(7\frac{3}{4}-2\frac{2}{5})\times\frac{5}{6}\times(\frac{4}{5}+\frac{1}{3})$.

NOTE.—The word *of* between fractions is equivalent to the sign of multiplication; and such an expression is sometimes called a **compound fraction**.

Find the values of the following indicated products:

27. $\frac{1}{3}$ of $\frac{4}{5}$ of $\frac{5}{6}$.
28. $\frac{3}{8}$ of $\frac{7}{8}$ of $\frac{8}{21}$.
29. $\frac{5}{8}$ of $\frac{4}{15}$ of $\frac{6}{7}$.
30. $\frac{11}{12}$ of $\frac{5}{33}$ of $\frac{18}{25}$ of $\frac{21}{23}$.
31. $\frac{1}{2}$ of $\frac{2}{3}$ of $\frac{3}{4}$ of $\frac{4}{5}$ of $\frac{5}{6}$ of $\frac{6}{7}$ of $\frac{7}{8}$ of $\frac{8}{9}$ of $\frac{9}{10}$.

NOTE. In the following examples, cancellation may be employed by the aid of the Factor Table, pages 110, 111.

32. $\frac{891}{715}\times\frac{1219}{1848}\times\frac{2448}{6868}$.
33. $\frac{2281}{2587}\times\frac{3881}{8408}\times\frac{4887}{6459}$.
34. $\frac{7379}{6061}\times\frac{8198}{8418}\times\frac{8549}{9797}$.
35. What will 7 cords of wood cost at \$ $3\frac{5}{8}$ per cord?
36. What is the value of $(\frac{3}{4})^2\times\frac{16}{21}\times(\frac{1}{5})^5$?
37. If a horse eats $\frac{3}{7}$ of a bushel of oats in a day, how many bushels will 10 horses eat in 6 days?
38. What is the cube of $12\frac{7}{8}$?

[redacted]

39. At \$ $9\frac{3}{5}$ per ton, what will be the cost of $\frac{1}{2}$ of $\frac{5}{6}$ of a ton of hay?

40. At \$ $\frac{9}{16}$ a bushel, what will be the cost of $1\frac{3}{5}$ bushels of corn?

41. A man owning $\frac{7}{8}$ of a farm, sold $\frac{1}{3}$ of his share. What part of the whole farm had he left?

42. A man bought a horse for \$ $125\frac{3}{4}$, and sold him for $\frac{4}{5}$ of what he cost. What was the loss?

43. A man owned $\frac{2}{3}$ of $123\frac{5}{6}$ acres of land, and sold $\frac{3}{5}$ of his share. How many acres did he sell?

44. If a family consumes $1\frac{1}{4}$ barrels of flour a month, how many barrels will five such families consume in $4\frac{9}{10}$ months?

45. When peaches are worth \$ $\frac{7}{8}$ per basket, what is $\frac{4}{7}$ of a basket worth?

46. A man owning $\frac{4}{5}$ of $156\frac{3}{8}$ acres of land, sold $\frac{1}{2}$ of $\frac{3}{4}$ of his share. How many acres did he sell?

47. What is the product of $(\frac{3}{8})^3 \times (\frac{1}{2})^2 \times (\frac{9}{10})^2 \times (3\frac{1}{3})^4$?

48. If a family consumes $1\frac{7}{8}$ barrels of flour a month, how many barrels will 6 families consume in $8\frac{9}{10}$ months?

49. What is the product of $150\frac{1}{2} - \{(\frac{6}{7} \text{ of } 121\frac{4}{5} + \frac{3}{4} \text{ of } 48\frac{2}{3}) - 75\}$ multiplied by $3 \times \{(\frac{4}{5} \text{ of } 1\frac{1}{8} \times 4) - 2\frac{1}{4}\}$?

50. What must be paid for $\frac{4}{5}$ of $6\frac{1}{2}$ tons of coal at $\frac{2}{3}$ of \$ $7\frac{1}{4}$ a ton?

51. A man at his death left his wife \$12500, which was $\frac{1}{2}$ of $\frac{5}{6}$ of his estate; she at her death left $\frac{5}{7}$ of her share to her daughter. What part of the father's estate did the daughter receive?

52. A owned $\frac{5}{6}$ of a cotton factory, and sold $\frac{3}{4}$ of his share to B, who sold $\frac{1}{2}$ of what he bought to C, who sold $\frac{2}{3}$ of what he bought to D. What part of the whole factory did each then own?

53. What is the value of:

$$\overline{2\frac{1}{4} \times \frac{1}{5}} + \frac{5}{6} \text{ of } 4\frac{1}{4} \times (\tfrac{2}{5})^2 + \overline{(3\tfrac{2}{3})^3 - (3\tfrac{2}{3})^2}\,?$$

DIVISION.

Examples.

263. To divide fractions by integers, integers by fractions, or fractions by fractions.

1. Divide $\frac{21}{25}$ by 3.

FIRST OPERATION.

$\frac{21}{25} \div 3 = \frac{7}{25}$ *Ans.*

SECOND OPERATION.

$\frac{21}{25} \div 3 = \frac{21}{75} = \frac{7}{25}$ *Ans.*

SOLUTION. — In the first operation we divide the fraction by 3 by dividing its numerator by 3, and in the second operation we divide the fraction by 3 by multiplying its denominator by 3 (§ 260, II or III).

2. Divide 15 by $\frac{3}{7}$.

FIRST OPERATION.

$15 \div \frac{3}{7} = 5 \times 7 = 35$ *Ans.*

SECOND OPERATION.

$15 \div \frac{3}{7} = 105 \div 3 = 35$ *Ans.*

SOLUTION. — To divide by $\frac{3}{7}$, we must divide by 3 and multiply by 7 (§ 261, II or III).

In the first operation, we first divide 15 by 3, and then multiply the quotient by 7.

In the second operation we first multiply 15 by 7, and then divide the product by 3.

3. Divide $\frac{4}{15}$ by $\frac{3}{5}$.

FIRST OPERATION.

1st step, $\frac{4}{15} \div 3 = \frac{4}{45}$

2d step, $\frac{4}{45} \times 5 = \frac{20}{45} = \frac{4}{9}$ *Ans.*

SECOND OPERATION.

$\frac{4}{15} \times \frac{5}{3} = \frac{20}{45} = \frac{4}{9}$ *Ans.*

THIRD OPERATION.

$$\frac{4}{\underset{3}{\cancel{15}}} \times \frac{\cancel{5}}{3} = \frac{4}{9} \text{ Ans.}$$

SOLUTION. — To divide by $\frac{3}{5}$, we must divide by 3 and multiply by 5 (§ 261, II or III). In the first operation we first divide $\frac{4}{15}$ by 3 by multiplying the denominator by 3. We then multiply the result, $\frac{4}{45}$ by 5, by multiplying the numerator by 5, giving $\frac{20}{45} = \frac{4}{9}$ for the required quotient. By inspecting this operation, we observe that the result, $\frac{20}{45}$, is obtained by multiplying the denominator of the given dividend by the numerator of the divisor, and the numerator of the dividend by the denominator of the divisor.

Hence in the second operation, we invert the terms of the divisor, $\frac{3}{5}$, and then find the product of the upper terms for a numerator, and of the lower terms for a denominator, and we obtain the same result as in the first operation. In the third operation, we shorten the process by cancellation.

NOTE. — We have learned (§ 160) that the reciprocal of a number is 1 divided by the number. If we divide 1 by $\frac{3}{5}$, we shall have $1 \div \frac{3}{5} = 1 \times \frac{5}{3} = \frac{5}{3}$. Hence, the reciprocal of a fraction is the fraction inverted.

RULE.—I. *Reduce integers and mixed numbers to improper fractions.*

II. *Multiply the dividend by the reciprocal of the divisor.*

NOTES.—1. If the vertical line is used, the numerators of the dividend and the denominators of the divisor must be written on the right of the vertical.

2. Since a compound fraction is an indicated product of several fractions, its reciprocal may be obtained by inverting each factor of the compound fraction.

4. Divide $\frac{3}{5}$ of $\frac{5}{9}$ by $\frac{7}{9}$ of $\frac{5}{14}$.

OPERATION.

$\frac{3}{5} \times \frac{5}{9} = \frac{1}{3}$

$\frac{7}{9} \times \frac{5}{14} = \frac{5}{18}$

$\frac{1}{3} \times \frac{18}{5} = \frac{6}{5} = 1\frac{1}{5}$ *Ans.*

Or,

$$\frac{3}{\cancel{5}} \times \frac{\cancel{5}}{\cancel{9}} \times \frac{\cancel{9}}{\cancel{7}} \times \frac{\overset{2}{\cancel{14}}}{5} = 1\frac{1}{5} \textit{ Ans.}$$

SOLUTION.—The dividend, reduced to a simple fraction, is $\frac{1}{3}$; the divisor, reduced in like manner, is $\frac{5}{18}$; and $\frac{1}{3}$ divided by $\frac{5}{18}$ is $1\frac{1}{5}$, the quotient required. Or, we may apply the general rule directly by inverting both factors of the divisor.

NOTE.—The second method of solution given above has two advantages: first, it gives the answer by a single operation; second, it affords greater facility for cancellation.

Divide:

5. $\frac{12}{35}$ by 4.
6. $\frac{10}{11}$ by 5.
7. $\frac{128}{135}$ by 80.
8. 10 by $\frac{2}{7}$.
9. 28 by $\frac{3}{4}$.
10. 56 by $1\frac{5}{9}$.
11. $\frac{15}{24}$ by $\frac{5}{6}$.
12. 3 by $\frac{5}{12}$.
13. $\frac{16}{27}$ by $\frac{8}{9}$.
14. $\frac{12}{55}$ by $\frac{9}{77}$.
15. $3\frac{5}{6}$ by $5\frac{1}{9}$.
16. $1\frac{14}{91}$ by $\frac{35}{52}$.
17. $\frac{4}{9}$ of $\frac{5}{11}$ by $\frac{8}{11}$ of $\frac{5}{18}$.
18. $\frac{7}{21}$ of $\frac{8}{18}$ by $\frac{7}{9}$ of $\frac{5}{21}$.
19. $2\frac{1}{2} \times 7\frac{1}{5}$ by $3\frac{1}{8} \times 3\frac{3}{10}$.
20. 11 by $\frac{2}{3} \times 5\frac{1}{2} \times 7$.

21. What is the value of $\dfrac{5\frac{1}{2}}{4\frac{2}{5}}$?

OPERATION.

$$\frac{5\frac{1}{2}}{4\frac{2}{5}} = \frac{\frac{11}{2}}{\frac{22}{5}} = \frac{\cancel{11}}{2} \times \frac{5}{\underset{2}{\cancel{22}}} = \frac{5}{4} = 1\frac{1}{4} \textit{ Ans.}$$

SOLUTION.—The fractional form indicates division, the numerator being the dividend and the denominator the divisor (§ 239, II); hence, we reduce the mixed numbers to improper fractions, and then treat the denominator, $\frac{22}{5}$, as a divisor, and obtain the result, $1\frac{1}{4}$, by the general rule for division of fractions.

NOTES. — 1. Expressions like $\frac{5\frac{1}{4}}{4\frac{2}{3}}$ and $\frac{\frac{11}{4}}{\frac{12}{5}}$ are sometimes called **complex fractions.**

2. In the reduction of complex fractions to simple fractions, if either the numerator or denominator consists of one or more parts connected by + or −, the operations indicated by these signs must first be performed, and afterward the division.

22. What is the value of $\frac{\frac{3}{4}}{\frac{7}{8}}$?

23. What is the value of $\frac{\frac{2}{3} \times \frac{11}{12}}{\frac{1}{18} \times 5\frac{1}{2}}$?

24. What is the value of $\frac{7 + 3\frac{5}{8}}{1\frac{5}{12}}$?

25. Reduce $\frac{\frac{3}{4} - \frac{3}{5}}{\frac{1}{3} + \frac{3}{8}}$ to its simplest form.

26. Reduce $\frac{\frac{5}{7} - \frac{2}{3}}{\frac{1}{3} \times \frac{2}{7}}$ to its simplest form.

27. Reduce $\frac{\frac{5}{9} \text{ of } \frac{3}{7}}{6\frac{1}{5} - 5\frac{4}{15}}$ to its simplest form.

28. Divide $720 - (\frac{5}{8} \times \overline{28 - 7\frac{1}{2}})$ by $\overline{40\frac{1}{4} + (\frac{9}{10} \div \frac{3}{5})} \times (\frac{1}{2})^4$.

29. What number is that which, if multiplied by $\frac{3}{4}$ of $\frac{5}{9}$ of $\frac{1}{3}$ of 9, will produce 48?

30. If 7 pounds of coffee cost \$ $\frac{4}{5}$, how much will 1 pound cost?

31. If a boy earns \$ $\frac{3}{8}$ a day, how many days will it take him to earn \$ $6\frac{1}{2}$?

32. If $\frac{4}{9}$ of an acre of land costs \$ 30, what will an acre cost at the same rate?

33. At $\frac{1}{2}$ of $\frac{3}{4}$ of a dollar a pint, how much oil can be bought for \$ $\frac{9}{10}$?

34. I bought $\frac{1}{5}$ of $4\frac{1}{8}$ cords of wood for $\frac{2}{5}$ of $\frac{1}{3}$ of \$ 30. How much was 1 cord worth at the same rate?

35. The product of two numbers is 27, and one of them is $2\frac{5}{9}$. What is the other?

36. By what number must you multiply $16\frac{11}{12}$ to produce $148\frac{3}{4}$?

37. What number is that which, if multiplied by $\frac{3}{8}$ of $\frac{5}{8}$ of 2, will produce $\frac{7}{9}$?

FRACTIONAL RELATION OF NUMBERS.

264. If we wish to know what part of 6 the number 3 is, we divide 3, the number to be compared, by 6, the number with which it is compared, and our answer is the required part, $\frac{3}{6}$, or $\frac{1}{2}$. In finding what part one number is of another, whether the numbers are integral or fractional, the number compared always represents a dividend, the number with which it is compared a divisor, and the part to be found a quotient.

Examples.

265. To find what part one number is of another.

1. What part of 36 is 28?

OPERATION.

$28 \div 36 = \frac{28}{36} = \frac{7}{9}$ *Ans.*

SOLUTION. — The number compared, 28, represents a dividend, and 36 the number with which it is compared, a divisor. $28 \div 36 = \frac{7}{9}$.

2. What part of 8 is $\frac{4}{5}$?

OPERATION.

$\frac{4}{5} \div 8 = \frac{4}{5} \times \frac{1}{8} = \frac{1}{10}$ *Ans.*

SOLUTION. — The number compared, $\frac{4}{5}$, divided by $8 = \frac{1}{10}$.

3. What is the relation of 8 to $\frac{4}{5}$?

OPERATION.

$8 \div \frac{4}{5} = 8 \times \frac{5}{4} = 10$ *Ans.*

SOLUTION. — The number compared, 8, divided by $\frac{4}{5}$, the number with which it is compared, = 10.

4. What part of $\frac{4}{5}$ is $\frac{2}{3}$?

OPERATION.

$\frac{2}{3} \div \frac{4}{5} = \frac{2}{3} \times \frac{5}{4} = \frac{5}{6}$ *Ans.*

SOLUTION. — In this example $\frac{2}{3}$ is the number compared; $\frac{2}{3} \div \frac{4}{5} = \frac{5}{6}$.

5. What part of $2\frac{4}{5}$ is $1\frac{2}{3}$?

OPERATION.

$2\frac{4}{5} = \frac{14}{5}$; $1\frac{2}{3} = \frac{5}{3}$

$\frac{5}{3} \div \frac{14}{5} = \frac{5}{3} \times \frac{5}{14} = \frac{25}{42}$ *Ans.*

SOLUTION. — We first reduced the mixed numbers to improper fractions, $\frac{14}{5}$ and $\frac{5}{3}$; $\frac{5}{3}$, the number compared, divided by $\frac{14}{5} = \frac{25}{42}$.

RULE. — *Divide the number compared by the number with which it is compared.*

What part of:

6. 18 is 16? 7. 39 is 13? 8. 84 is 20? 9. 20 is 8? 10. 100 is 5?

11. 16 is $\frac{8}{9}$? 12. 28 is $\frac{7}{11}$? 13. 48 is $\frac{6}{18}$? 14. 15 is $\frac{3}{8}$? 15. 50 is $\frac{5}{8}$?

16. $\frac{4}{7}$ is $\frac{1}{2}$? 17. $\frac{3}{8}$ is $\frac{2}{9}$? 18. $\frac{4}{11}$ is $\frac{3}{7}$? 19. $\frac{5}{6}$ is $\frac{2}{3}$? 20. $\frac{8}{9}$ is $\frac{3}{5}$?

21. $2\frac{1}{6}$ is $1\frac{1}{4}$? 22. $9\frac{1}{3}$ is $3\frac{1}{7}$? 23. $25\frac{1}{7}$ is $6\frac{1}{4}$? 24. 100 is $33\frac{1}{3}$? 25. 50 is $12\frac{1}{2}$?

What is the relation of:

26. 5 to $\frac{3}{4}$? 27. 6 to $\frac{4}{5}$? 28. 14 to $\frac{35}{49}$?

29. $\frac{8}{9}$ to 3? 30. $\frac{4}{5}$ to 6? 31. $\frac{1}{3}$ to 7?

32. $\frac{3}{7}$ to $\frac{4}{7}$? 33. $\frac{1}{9}$ to $\frac{5}{8}$? 34. $\frac{4}{5}$ to $\frac{4}{9}$?

35. $3\frac{1}{3}$ to $4\frac{1}{2}$? 36. $6\frac{1}{3}$ to $2\frac{1}{4}$? 37. 1 to $5\frac{1}{2}$?

266. To find a number when a fractional part of it is given.

1. 180 is $\frac{3}{4}$ of what number?

OPERATION.

$\frac{1}{4}$ the number $= \frac{1}{3}$ of $180 = 60$
$4 \times 60 = 240$ *Ans.* Or,

$$\frac{4}{\cancel{3}} \text{ of } \overset{60}{\cancel{180}} = 240 \quad \textit{Ans.}$$

SOLUTION.—Since 180 is *three* fourths of a certain number, *one* fourth of the number will be $\frac{1}{3}$ of 180, which is 60; and the number will be 4 times 60, which is 240. Hence, 180 is $\frac{3}{4}$ of 240.

Or, $\frac{1}{4}$ the number will be $\frac{1}{3}$ of 180, and $\frac{4}{4}$, or the whole number, will be 4 times $\frac{1}{3}$ or $\frac{4}{3}$ of 180, which is 240.

2. $\frac{12}{15}$ is $\frac{3}{4}$ of what number?

OPERATION.

$$\frac{4}{\cancel{3}} \text{ of } \frac{\overset{4}{\cancel{12}}}{15} = \frac{16}{15} = 1\frac{1}{15} \quad \textit{Ans.}$$

SOLUTION.—$\frac{1}{4}$ the number will be $\frac{1}{3}$ of $\frac{12}{15}$ and $\frac{4}{4}$ or the whole number will be $\frac{4}{3}$ of $\frac{12}{15}$ $= 1\frac{1}{15}$ *Ans.*

RULE.—*Multiply the number by the fraction with its terms inverted; or, divide the number by the fraction.*

Find the number of which:

3. 75 is $\frac{5}{7}$. 4. 84 is $\frac{12}{20}$. 5. 50 is $\frac{5}{6}$. 6. 49 is $\frac{7}{8}$.

7. $15\frac{3}{4}$ is $\frac{3}{7}$. 8. 180 is $\frac{3}{4}$. 9. 240 is $\frac{4}{9}$. 10. 1000 is $\frac{10}{11}$.

11. 392 is $\frac{4}{9}$. 12. $\frac{12}{17}$ is $\frac{4}{5}$. 13. $6\frac{1}{2}$ is $\frac{1}{8}$. 14. $\frac{3}{4}$ is $\frac{1}{8}$.

15. 625 is $\frac{5}{6}$. 16. 450 is $\frac{5}{8}$. 17. $15\frac{1}{3}$ is $\frac{2}{3}$. 18. $\frac{3}{4}$ is $\frac{1}{3}$.

CONTINUED FRACTIONS.

267. If we take any fraction in its lowest terms, as $\frac{13}{54}$, and divide both terms by the numerator, we shall obtain a complex fraction, thus:

$$\frac{13}{54}=\cfrac{1}{4+\cfrac{2}{13}}$$

Reducing $\frac{2}{13}$, the fractional part of the denominator, in the same manner, we have,

$$\frac{13}{54}=\cfrac{1}{4+\cfrac{1}{6+\cfrac{1}{2}}}$$

Expressions in this form are called *continued fractions.*

268. A **Continued Fraction** is a fraction whose numerator is 1, and whose denominator is a whole number plus a fraction whose numerator is also 1, and whose denominator is a whole number plus a fraction, and so on.

269. The **Terms** of a continued fraction are the several simple fractions which form the parts of the continued fraction; as, $\frac{1}{4}$, $\frac{1}{6}$, and $\frac{1}{2}$.

NOTE. — Continued fractions are sometimes written with the sign of addition between the denominators, thus: $\frac{1}{4+}\frac{1}{6+}\frac{1}{2}$

270. An **Approximate Value** of a continued fraction is the simple fraction obtained by reducing one, two, three, or more terms of the continued fraction.

271. To reduce any fraction to a continued fraction.

1. Reduce $\frac{109}{339}$ to a continued fraction.

OPERATION.

$$\frac{109}{339}=\cfrac{1}{3+\cfrac{1}{9+\cfrac{1}{12}}}$$

SOLUTION. — We divide the denominator, 339, by the numerator, 109, and obtain 3 for the denominator of the first term of the continued fraction. Then in the same manner we divide the last divisor, 109, by the remainder, 12, and obtain 9 for the denominator of the second term of the continued fraction. In like manner we obtain 12 for the denominator of the final term.

RULE. — I. *Divide the greater term by the less, and the last divisor by the last remainder, and so on, till there is no remainder.*

II. *Write* 1 *for the numerator of each term of the continued fraction, and the quotients in their order for the denominators.*

Reduce to continued fractions:

2. $\frac{259}{1795}$. 3. $\frac{1249}{6721}$. 4. $\frac{39}{121}$. 5. $\frac{222874}{516601}$.

272. To find the several approximate values of a continued fraction.

1. Reduce $\frac{38}{163}$ to a continued fraction, and find its approximate values.

OPERATION.

$$\frac{38}{163}=\cfrac{1}{4+\cfrac{1}{3+\cfrac{1}{2+\cfrac{1}{5}}}}, \text{ the continued fraction.}$$

$$\frac{1}{4} = \frac{1}{4}, \text{ 1st approx. value.}$$

$$\cfrac{1}{4+\cfrac{1}{3}} = \frac{3}{4\times3+1} = \frac{3}{13}, \text{ 2d " "}$$

$$\cfrac{1}{4+\cfrac{1}{3+\cfrac{1}{2}}} = \cfrac{1}{4+\cfrac{2}{3\times2+1}} = \frac{3\times2+1}{(4\times3+1)\times2+4} = \frac{3\times2+1}{13\times2+4} = \frac{7}{30}, \text{ 3d " "}$$

$$\cfrac{1}{4+\cfrac{1}{3+\cfrac{1}{2+\cfrac{1}{5}}}} = \frac{7\times5+3}{30\times5+13} = \frac{38}{163}, \text{ 4th " "}$$

SOLUTION. — We take $\frac{1}{4}$, the first term of the continued fraction, for the first approximate value. Reducing the complex fraction formed by the first two terms of the continued fraction, we have $\frac{3}{13}$ for the second approximate value. In like manner, reducing the first three terms, we have $\frac{7}{30}$ for the third approximate value. By examining this last process, we perceive that the third approximate value, $\frac{7}{30}$, is obtained by multiplying the terms of the preceding approximation, $\frac{3}{13}$, by the denominator of the third term of the continued fraction, 2, and adding the corresponding terms of the first approximate value. Taking advantage of this principle, we multiply the terms of $\frac{7}{30}$ by the fourth denominator, 5, in the con-

tinued fraction, and adding the corresponding terms of $\frac{3}{13}$, obtain $\frac{38}{163}$, the fourth approximate value, which is the same as the original fraction.

RULE. — I. *For the first approximate value, take the first term of the continued fraction.*

II. *For the second approximate value, reduce the complex fraction formed by the first two terms of the continued fraction.*

III. *For each succeeding approximate value, multiply both numerator and denominator of the last preceding approximation by the next denominator in the continued fraction, and add to the corresponding products respectively the numerator and denominator of the preceding approximation.*

NOTES. — 1. When the given fraction is improper, invert it, and reduce this result to a continued fraction; then invert the approximate values obtained therefrom.

2. In a series of approximate values, the 1st, 3d, 5th, etc., are greater than the given fraction; and the 2d, 4th, 6th, etc., are less than the given fraction.

We may also reduce a continued fraction to a common fraction by beginning at the last term.

OPERATION.

$$\cfrac{1}{2+\cfrac{1}{5}}=\frac{5}{11}$$

$$\cfrac{1}{3+\cfrac{5}{11}}=\frac{11}{38}$$

$$\cfrac{1}{4+\cfrac{11}{38}}=\frac{38}{163}\quad \textit{Ans.}$$

SOLUTION. — Taking the same example, and reducing the last two terms, we have $1 \div \frac{11}{5} = \frac{5}{11}$. Taking this result with the preceding term, we have $\cfrac{1}{3+\cfrac{5}{11}} = 1 \div \frac{38}{11} = \frac{11}{38}$. Again, taking this result with the first term, we have $\cfrac{1}{4+\cfrac{11}{38}} = 1 \div \frac{163}{38} = \frac{38}{163}$.

2. Find the approximate values of $\frac{67}{155}$.

3. Find the approximate values of $\frac{83}{849}$.

4. What are the first three approximate values of $\frac{2831}{20857}$?

5. What are the first five approximate values of $\frac{227}{732}$?

6. Reduce $\frac{29}{42}$ to the form of a continued fraction, and find the value of each approximating fraction.

Reduce to common fractions:

7. $\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{4}$. 8. $\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}$. 9. $\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}$.

GREATEST COMMON DIVISOR.

273. The **Greatest Common Divisor** of two or more fractions is the greatest number which will exactly divide each of them, giving a whole number for a quotient.

NOTE.—The definition of greatest common divisor (§ 219) is general, and applies to fractions as well as to integers.

274. In the division of one fraction by another the quotient will be a whole number, if, when the divisor is inverted, the two lower terms may both be canceled. This will be the case when the numerator of the divisor is exactly contained in the numerator of the dividend, and the denominator of the divisor exactly contains, *or is a multiple of,* the denominator of the dividend; as,

$$\frac{10}{13} \div \frac{5}{39} = \frac{\overset{2}{\cancel{10}}}{\cancel{13}} \times \frac{\overset{3}{\cancel{39}}}{\cancel{5}} = 6$$

PRINCIPLES.—I. *A fraction is an exact divisor of a given fraction when its numerator is a divisor of the numerator of the given fraction, and its denominator is a multiple of the denominator of the given fraction.*

II. *A fraction is a common divisor of two or more given fractions when its numerator is a common divisor of the numerators of the given fractions, and its denominator is a common multiple of the denominators of the given fractions.*

III. *The greatest common divisor of two or more given fractions is a fraction whose numerator is the greatest common divisor of the numerators of the given fractions, and whose denominator is the least common multiple of the denominators of the given fractions.*

The greatest common divisor of two or more fractions may be found by reducing the fractions to a common denominator, and then finding the greatest common divisor of the numerators of the similar fractions for a numerator, which is written over the common denominator. The following method, however, which is developed directly from the principles, is shorter:

Examples.

275. To find the greatest common divisor of two or more fractions.

1. What is the greatest common divisor of $\frac{5}{6}$, $\frac{5}{12}$, and $\frac{15}{16}$?

SOLUTION.—The greatest common divisor of 5, 5, and 15, the given numerators, is 5. The least common multiple of 6, 12, and 16, the given denominators, is 48. Therefore the greatest common divisor of the given fractions is $\frac{5}{48}$ (§ 274, III).

PROOF.

$$\left.\begin{array}{l} \frac{5}{6} \div \frac{5}{48} = \frac{\cancel{5}}{\cancel{6}} \times \frac{\overset{8}{\cancel{48}}}{\cancel{5}} = 8 \\[2ex] \frac{5}{12} \div \frac{5}{48} = \frac{\cancel{5}}{\cancel{12}} \times \frac{\overset{4}{\cancel{48}}}{\cancel{5}} = 4 \\[2ex] \frac{15}{16} \div \frac{5}{48} = \frac{\overset{3}{\cancel{15}}}{\cancel{16}} \times \frac{\overset{3}{\cancel{48}}}{\cancel{5}} = 9 \end{array}\right\} \textit{Prime} \text{ to each other.}$$

RULE.—*Find the greatest common divisor of the given numerators for a new numerator, and the least common multiple of the given denominators for a new denominator. This fraction will be the greatest common divisor sought.*

NOTE.—Whole and mixed numbers must first be reduced to improper fractions, and all fractions to their lowest terms.

What is the greatest common divisor of:

2. $\frac{7}{9}$, $\frac{14}{27}$, and $\frac{28}{45}$?

3. $3\frac{1}{5}$, $1\frac{5}{7}$, and $\frac{24}{35}$?

4. 4, $2\frac{2}{9}$, $2\frac{2}{5}$, and $\frac{4}{90}$?

5. $109\frac{1}{5}$ and $122\frac{4}{7}$?

6. What is the length of the longest measure that can be exactly contained in each of the two distances, $18\frac{2}{5}$ feet and $57\frac{1}{2}$ feet?

7. A merchant has three kinds of oil; of the first $134\frac{3}{4}$ gallons, of the second $128\frac{1}{8}$ gallons, of the third $115\frac{1}{2}$ gallons; he wishes to ship the same in full casks of equal size. What is the least number he can use without mixing the different kinds of oil? How many casks will be required?

LEAST COMMON MULTIPLE.

276. The **Least Common Multiple** of two or more fractions is the least number which can be exactly divided by each of them, giving a whole number for a quotient.

NOTE. — It is evident from this that the definition of least common multiple, § 225, is general, and applies to fractions as well as to integers.

277. Since in performing operations in division of fractions the divisor is inverted, it is evident that one fraction will exactly contain another when the numerator of the dividend exactly contains the numerator of the divisor, and the denominator of the dividend is exactly contained in the denominator of the divisor; as,

$$\frac{9}{25} \div \frac{3}{75} = \frac{\overset{3}{\cancel{9}}}{\cancel{25}} \times \frac{\overset{3}{\cancel{75}}}{\cancel{3}} = 9.$$

Hence, $\frac{9}{25}$ is a multiple of $\frac{3}{75}$.

PRINCIPLES. — I. *A fraction is a multiple of a given fraction when its numerator is a multiple of the numerator of the given fraction, and its denominator is a divisor of the denominator of the given fraction.*

II. *A fraction is a common multiple of two or more given fractions when its numerator is a common multiple of the numerators of the given fractions, and its denominator is a common divisor of the denominators of the given fractions.*

III. *The least common multiple of two or more given fractions is a fraction whose numerator is the least common multiple of the numerators of the given fractions, and whose denominator is the greatest common divisor of the denominators of the given fractions.*

NOTE. — The least whole number that will exactly contain two or more given fractions in their lowest terms, is the least common multiple of their numerators.

The least common multiple of two or more fractions may be found by reducing them to a common denominator and finding the least common multiple of the numerators of the similar fractions, which is written over the common denominator; but the following method, which is directly derived from the principles, is shorter:

Examples.

278. To find the least common multiple of two or more fractions.

1. What is the least common multiple of $\frac{3}{4}$, $\frac{5}{12}$, and $\frac{15}{16}$?

SOLUTION.—The least common multiple of 3, 5, and 15, the given numerators, is 15; the greatest common divisor of 4, 12, and 16, the given denominators, is 4. Hence, the least common multiple of the given fractions is $\frac{15}{4} = 3\frac{3}{4}$ (§ 277, III).

PROOF.

$$\left.\begin{aligned}
\frac{15}{4} \div \frac{3}{4} &= \frac{\overset{5}{\cancel{15}}}{\cancel{4}} \times \frac{\cancel{4}}{\cancel{3}} = 5 \\
\frac{15}{4} \div \frac{5}{12} &= \frac{\overset{3}{\cancel{15}}}{\cancel{4}} \times \frac{\overset{3}{\cancel{12}}}{\cancel{5}} = 9 \\
\frac{15}{4} \div \frac{15}{16} &= \frac{\cancel{15}}{\cancel{4}} \times \frac{\overset{4}{\cancel{16}}}{\cancel{15}} = 4
\end{aligned}\right\} \textit{Prime} \text{ to each other.}$$

RULE.—*Find the least common multiple of the given numerators for a new numerator, and the greatest common divisor of the given denominators for a new denominator. This fraction will be the least common multiple sought.*

NOTE.—Mixed numbers and integers should be reduced to improper fractions, and all fractions to their lowest terms, before applying the rule.

2. What is the least common multiple of $\frac{2}{5}$, $\frac{7}{10}$, $\frac{14}{15}$, and $\frac{8}{25}$?

3. What is the least common multiple of $\frac{7}{24}$, $\frac{35}{36}$, and $\frac{49}{60}$?

4. What is the least common multiple of $2\frac{22}{25}$, $1\frac{37}{75}$, and $\frac{63}{100}$?

5. What is the least common multiple of $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$, $\frac{4}{5}$, $\frac{5}{6}$, $\frac{6}{7}$, $\frac{7}{8}$, $\frac{8}{9}$, and $\frac{9}{10}$?

6. What is the least common multiple of $\frac{1}{2}$, $\frac{1}{4}$, $\frac{3}{5}$, $\frac{4}{7}$, $\frac{5}{8}$, $\frac{2}{3}$, $\frac{1}{6}$, and $\frac{1}{10}$?

7. The driving wheels of a locomotive are $15\frac{5}{16}$ feet in circumference, and the trucks $9\frac{3}{8}$ feet in circumference. What distance must the train move, in order to bring the wheel and truck in the same relative positions as at starting?

Examples for Review.

279. 1. Change $\frac{7}{9}$ of $\frac{5}{7}$ to an equivalent fraction having 135 for its denominator.

2. Reduce $\frac{3}{4}$, $\frac{1}{6}$, $\frac{5}{8}$, and $\frac{11}{12}$ to equivalent fractions, whose denominators shall be 48.

3. Find the least common denominator of $1\frac{1}{3}$, $\frac{6}{7}$, 2, $\frac{7}{10}$, $\frac{2}{3}$ of $\frac{4}{5}$, $\frac{4}{9}$ of $\frac{1}{4}$.

4. The sum of $\dfrac{\frac{2}{3} \text{ of } \frac{3}{4}}{\frac{1}{2}}$ and $\dfrac{\frac{2}{5} \text{ of } \frac{5}{6}}{\frac{2}{9} \text{ of } 4\frac{1}{2}}$ is equal to how many times their difference?

5. The less of two numbers is $\dfrac{54\frac{3}{5}}{\frac{1}{5} \text{ of } 8\frac{2}{3}}$, and their difference $\dfrac{1\frac{5}{9}}{\frac{16}{9}}$. What is the greater number?

6. What number multiplied by $\frac{2}{9}$ of $\frac{5}{8} \times 3\frac{2}{7}$, will produce $\frac{23}{84}$?

7. Find the value of:

$$\frac{2-\frac{1}{4}}{2} \times \frac{(8\frac{4}{7})^2}{12} + \overline{(2+\frac{1}{5}) \div (3+\frac{1}{7})} + \frac{11\frac{3}{32}}{8\frac{7}{8}}.$$

8. What number diminished by the difference between $\frac{2}{7}$ and $\frac{7}{9}$ of itself, leaves a remainder of 144?

9. A man after spending $\frac{1}{3}$, $\frac{2}{5}$, and $\frac{1}{8}$ of his money, had $119 left. How much had he at first?

10. What will $\frac{1}{2}$ of $10\frac{7}{8}$ cords of wood cost, at $\frac{4}{29}$ of $42 per cord?

11. There are two numbers whose difference is $25\frac{7}{15}$, and one number is $\frac{5}{7}$ of the other. What are the numbers?

12. Divide $2000 between A and B, so that one shall have $\frac{7}{9}$ as much as the other.

Suggestion. — There will be $7 + 9 = 16$ parts. A will receive $\frac{7}{16}$ and B $\frac{9}{16}$.

13. A horse and carriage cost $700. If the carriage cost $\frac{2}{5}$ as much as the horse, what was the cost of each?

14. If a man travels 4 miles in $\frac{3}{5}$ of an hour, how far would he travel in $1\frac{1}{2}$ hours at the same rate?

15. At $\$\frac{7}{8}$ a yard, how many yards of silk can be bought for $\$10\frac{5}{8}$?

16. How many bushels of oats, worth $ $\frac{2}{5}$ a bushel, will pay for $\frac{2}{3}$ of a barrel of flour at $ $4\frac{1}{3}$ a barrel?

17. A boy having lost $\frac{1}{4}$ of his money and then $\frac{1}{3}$ of the remainder, had $ 2.50 left. How much had he at first?

SUGGESTION. — $\frac{3}{4} - (\frac{1}{3}$ of $\frac{3}{4}) =$ $ 2.50.

18. A man lost 20% of his money in business and used $\frac{1}{8}$ of the remainder for household expenses, after which he had $ 8000. How much had he at first?

19. If $\frac{2}{5}$ of a bushel of barley is worth $\frac{3}{4}$ of a bushel of corn, and corn is worth $ $\frac{3}{8}$ per bushel, how many bushels of barley can be bought for $ 15?

20. If 48 is $\frac{5}{7}$ of some number, what is $\frac{3}{4}$ of the same number?

21. A man divided his estate among his three sons as follows: to the first son he gave $\frac{3}{8}$ of it; to the second he gave $\frac{2}{3}$ of the remainder. The difference between the portions given to the first and second son was $ 500. What was the value of the whole estate, and how much was the third son's share?

22. If $7\frac{1}{2}$ tons of hay cost $ 60, how many tons can be bought for $ 78, at the same rate?

23. If a man agrees to do a job of work in 30 days, what part of it ought he to do in $16\frac{1}{2}$ days?

24. A can do a piece of work in 5 days, B can do it in 6 days, and C in 4 days. In what time can they do it working together?

SOLUTION. — We must first find what part each can do in 1 day. A can do $\frac{1}{5}$ of the work in 1 day, B $\frac{1}{6}$, C $\frac{1}{4}$. Together they do $\frac{1}{5} + \frac{1}{6} + \frac{1}{4} = \frac{37}{60}$ in 1 day. Therefore it will take them $1 \div \frac{37}{60} = \frac{60}{37} = 1\frac{23}{37}$ days *Ans.*

25. A can do a certain piece of work in 8 days, and B can do the same work in 6 days. In what time can both together do it?

26. A certain pipe will fill a cistern with water in 20 minutes, and another pipe will fill it in 15 minutes. If both pipes are running together, in what time will the cistern be filled?

27. A can shell a number of peas in 45 minutes, B in 30 minutes, C in 35 minutes, and D in 60 minutes. In what time can they shell them if they all work together?

28. A father divided a piece of land among his three sons; to the first he gave $12\frac{1}{4}$ acres, to the second $\frac{3}{8}$ of the whole, and to the third as much as to the other two. How many acres did the third have?

29. If cloth $1\frac{1}{4}$ yards in breadth requires $20\frac{1}{2}$ yards in length to make a certain number of garments, how many yards in length will cloth $\frac{3}{4}$ of a yard wide require to make the same number?

30. A man owning $\frac{5}{8}$ of an iron foundry, sold $\frac{4}{5}$ of his share for $2570\frac{2}{3}$. How much was the whole foundry worth?

31. Suppose the cargo of a vessel to be worth $10000, and $\frac{4}{9}$ of $\frac{5}{7}$ of $\frac{9}{10}$ of the vessel to be worth $\frac{1}{4}$ of $\frac{4}{5}$ of $1\frac{5}{7}$ of the cargo; what is the whole value of the ship and cargo?

32. If $\frac{3}{4}$ of 6 bushels of wheat cost $4\frac{1}{2}$, how much will $\frac{4}{5}$ of 1 bushel cost?

33. A man engaging in trade lost $\frac{2}{5}$ of the money he invested, after which he gained $740, and then had $3500. How much did he lose?

34. A cistern being full of water sprung a leak, and before it could be stopped, $\frac{5}{8}$ of the water ran out, but $\frac{2}{5}$ as much ran in at the same time. What part of the cistern was emptied?

35. A certain pipe will fill a tub in 8 minutes, and another pipe will empty it in 5 minutes. If the tub is full, and both pipes are opened, in what time will it be emptied?

Solution.—The first pipe fills $\frac{1}{8}$ every minute and the second pipe empties $\frac{1}{5}$ every minute. Hence, $\frac{1}{5}-\frac{1}{8}$ or $\frac{3}{40}$ is emptied every minute, and $1 \div \frac{3}{40} = \frac{40}{3} = 13\frac{1}{3}$ minutes will be required to empty the tub.

36. A merchant sold 5 barrels of flour for $32\frac{1}{2}$, which was $\frac{5}{8}$ as much as he received for all he had left, at $4 a barrel. How many barrels in all did he sell?

37. What is the least number of gallons of vinegar, expressed by a whole number, that will exactly fill, without waste, bottles containing either $\frac{5}{6}$, $\frac{3}{8}$, $\frac{6}{7}$, or $\frac{4}{5}$ gallons?

38. A, B, and C start at the same point in the circumference of a circular island, and travel round it in the same direction. A makes $\frac{2}{7}$ of a revolution in a day, B $\frac{4}{17}$, and C $\frac{8}{51}$. In how many days will they all be together at the point of starting?

39. Two men are $64\frac{3}{4}$ miles apart, and travel toward each other; when they meet one has traveled $5\frac{1}{2}$ miles more than the other. How far has each traveled?

40. There are two numbers whose sum is $1\frac{1}{10}$, and whose difference is $\frac{2}{5}$. What are the numbers?

41. A, B, and C own a ferry boat; A owns $\frac{13}{126}$ of the boat, and B owns $\frac{7}{18}$ of the boat more than C. What shares do B and C own respectively?

42. A schoolboy being asked how many dollars he had, replied, that if his money were multiplied by $\frac{14}{15}$, and $\frac{1}{6}$ of a dollar were added to the product, and $\frac{2}{5}$ of a dollar taken from the sum, this remainder divided by $\frac{6}{25}$ would be equal to the reciprocal of $\frac{4}{5}$ of a dollar. How much money had he?

43. A man bought 728 pounds of candles at $16\frac{3}{8}$ cents a pound. Had they been purchased for $3\frac{7}{8}$ cents less a pound, how many pounds could have been bought for the same money?

44. What number, divided by $1\frac{3}{8}$, will give a quotient of $9\frac{1}{8}$?

45. A man put his money into 4 packages; in the first he put $\frac{2}{5}$, in the second $\frac{1}{8}$, in the third $\frac{1}{6}$, and in the fourth the remainder, which was \$24 more than $\frac{1}{15}$ of the whole. How much money had he?

46. A boy lost $\frac{1}{2}$ of his kite string, and then added 30 feet, when it was just $\frac{4}{5}$ of its original length. What was the length at first?

47. A post stands $\frac{1}{6}$ in the mud, $\frac{1}{4}$ in the water, and 21 feet above the water. What is its length?

48. A cistern which is filled by a certain pipe in 3 hours, is emptied by another pipe in 4 hours. If the cistern is empty and both pipes are opened, how long will it take to fill it?

49. If the cistern, in Example 48, is full and both pipes are opened, what will be the amount of overflow in 1 hour?

50. If a certain number is increased by $1\frac{3}{4}$, this sum diminished by $\frac{3}{8}$, this remainder multiplied by $5\frac{2}{3}$, and this product divided by $1\frac{3}{7}$, the quotient will be $7\frac{1}{2}$. What is the number?

51. If $\frac{3}{8}$ of $\frac{4}{5}$ of $3\frac{1}{2}$ times 1 is multiplied by $\frac{7}{8}$, the product divided by $\frac{3}{5}$, the quotient increased by $4\frac{1}{6}$, and the sum diminished by $\frac{4}{7}$ of itself, what will the remainder be?

52. What number is that whose $\frac{1}{5}$ and $\frac{1}{6}$ together equal 9?

53. A and B together can do a piece of work in 2 days, A and C in 3 days, and B and C in 4 days. How long will it take them all together, and how long will it take each working alone to do the work?

SOLUTION. — A and B together do $\frac{1}{2}$ the work in 1 day.
A and C together do $\frac{1}{3}$ the work in 1 day.
B and C together do $\frac{1}{4}$ the work in 1 day.

$\frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{13}{12} =$ *twice* the work A, B, and C do in 1 day,
since A's + B's + A's + C's + B's + C's = 2 × (A's + B's + C's).
$\frac{13}{12} \div 2 = \frac{13}{24}$, the work A, B, and C do in 1 day.
$1 \div \frac{13}{24} = \frac{24}{13} = 1\frac{11}{13}$, the number of days in which A, B, and C together do the work.

Since A, B, and C do $\frac{13}{24}$ in 1 day, and A and B $\frac{1}{2}$ or $\frac{12}{24}$, C must do $\frac{13}{24} - \frac{12}{24} = \frac{1}{24}$ in 1 day, and it will take him 24 days to do the work. B must do $\frac{13}{24} - \frac{1}{3} = \frac{5}{24}$ in 1 day, and it will take him $4\frac{4}{5}$ days, and A must do $\frac{13}{24} - \frac{6}{24} = \frac{7}{24}$ in 1 day, and it will take him $3\frac{3}{7}$ days.

All together will do the work in $1\frac{11}{13}$ days.
A alone will do the work in $3\frac{3}{7}$ days.
B alone will do the work in $4\frac{4}{5}$ days.
C alone will do the work in 24 days. } *Ans.*

54. A, B, and C can do a piece of work together in 5 days, B and C can do it in 6 days, and A and C in 8 days. In what time can each do the work alone?

55. Two pipes, A and B, can fill a tank in 8 minutes, B and C can fill it in 10 minutes, and A and C in 12 minutes. In what time will the tank be filled if all three pipes are opened? In what time will each pipe alone fill the tank?

56. A can do a piece of work in 8 days, A and B can do it in 5 days, and B and C in 6 days. In what time can A, B, and C do the work? A and C? B? C?

57. If to $\frac{4}{5}$ of a certain number you add $\frac{3}{5}$ of $\frac{2}{3}$ of $\frac{5}{6}$ of 18, the answer will be 22. What is the number?

DECIMAL FRACTIONS.

280. A **Decimal Fraction** is one or more of the decimal divisions of a unit (§ 13).

NOTES.—1. The word *decimal* is derived from the Latin *decem*, which signifies *ten*.
2. Decimal fractions are commonly called *decimals*.

For Notation, Addition, and Subtraction of decimals, see §§ 47–58, 73, 86.

REDUCTION.

Examples.

281. To reduce decimals to a common denominator.

1. Reduce .5, .24, .7836, and .375 to a common denominator.

OPERATION.

$$\left.\begin{array}{r} .5000 \\ .2400 \\ .7836 \\ .3750 \end{array}\right\} Ans.$$

SOLUTION.—A common denominator must contain as many decimal places as are equal to the greatest number of decimal places in any of the given decimals. We find that the third number contains four decimal places, hence 10000 must be a common denominator. As annexing ciphers to decimals does not alter their value, we give to each number four decimal places, by annexing ciphers, and thus reduce the given decimals to a common denominator.

RULE.—*Give to each number the same number of decimal places, by annexing ciphers.*

NOTES.—1. If the numbers are reduced to the denominator of that one of the given numbers having the greatest number of decimal places, they will have their *least common decimal denominator*.

2. An integer may readily be reduced to any decimal by placing the decimal point after units, and annexing ciphers; one cipher reducing it to tenths, two ciphers to hundredths, three ciphers to thousandths, and so on.

Reduce to their least common denominator:

2. .18, .456, .0075, .000001, .05, .3789, .5943786, and .001.

3. 12 thousandths, 185 millionths, and 936 billionths.

4. 57.3, 900, 4.7555, and 100.000001.

282. To reduce a decimal to a common fraction.

1. Reduce .375 to an equivalent common fraction.

OPERATION.

$.375 = \frac{375}{1000} = \frac{3}{8}$

SOLUTION. — Writing the decimal figures 375 over the denominator, 1000, we have $\frac{375}{1000}$, which, reduced to its lowest terms, equals $\frac{3}{8}$.

RULE. — *Omit the decimal point, supply the proper denominator, and reduce the fraction to its lowest terms.*

Reduce to common fractions:

2. .752.	5. .68.	8. .375.	11. .00032.
3. .628.	6. .5625.	9. .625.	12. .002624.
4. .12.	7. .024.	10. .875.	13. .00315.

14. Reduce $.13\frac{1}{3}$ to a common fraction.

OPERATION.

$$.13\frac{1}{3} = \frac{13\frac{1}{3}}{100} = \frac{40}{300} = \frac{2}{15} \textit{ Ans.}$$

NOTE. — The decimal $.13\frac{1}{3}$ may properly be called a complex decimal.

Change to common fractions:

15. $.57\frac{1}{7}$.	17. $.14\frac{2}{7}$.	19. $.024\frac{3}{8}$.	21. $.444\frac{4}{9}$.
16. $.66\frac{2}{3}$.	18. $.42\frac{6}{7}$.	20. $.984\frac{3}{8}$.	22. .875.

23. Express 24.74 by an integer and a common fraction.

24. Reduce 2.1875 to an improper fraction.

25. Reduce 1.64 to an improper fraction.

26. Reduce 7.496 to an improper fraction.

283. To reduce a common fraction to a decimal.

1. Reduce $\frac{5}{8}$ to its equivalent decimal.

FIRST OPERATION.

$\frac{5}{8} = \frac{5000}{8000} = \frac{625}{1000} = .625$ *Ans.*

SECOND OPERATION.

$$\begin{array}{r} 8\underline{)5.000} \\ .625 \textit{ Ans.} \end{array}$$

SOLUTION. — We first annex the same number of ciphers to both terms of the fraction; this does not alter its value (§ 245, III); we then divide both resulting terms by 8, the significant figure of the denominator, and obtain the decimal denominator, 1000. Omitting the denominator, and prefixing the sign, we have the equivalent decimal, .625.

In the second operation, we omit the intermediate steps, and obtain the result, practically, by annexing the three ciphers to the numerator, 5, and dividing the result by the denominator, 8. Or, since $\frac{5}{8}$ is $\frac{1}{8}$ of 5, we divide 5 (reduced to a decimal) by 8.

2. Reduce $\frac{3}{125}$ to a decimal.

OPERATION.

$125)3.000$

$.024$ *Ans.*

SOLUTION. — Dividing as in Ex. 1, we obtain a quotient of 2 figures, 24. But since 3 ciphers have been annexed to the numerator, 3, there must be three places in the required decimal; hence we prefix 1 cipher to the quotient figures, 24. The reason of this is also shown as follows:

$$\frac{3}{125} = \frac{3000}{125000} = \frac{24}{1000} = .024$$

RULE. — *Annex ciphers to the numerator, and divide by the denominator. Point off as many decimal places in the result as are equal to the number of ciphers annexed.*

NOTE. — If the division is not exact when a sufficient number of decimal terms have been obtained, the sign, +, may be annexed to the decimal to indicate that there is still a remainder. When this remainder is such that the next decimal term would be 5 or greater than 5, the last term of the terminated decimal may be increased by 1, and the sign, –, annexed. And in general, + denotes that the written decimal is too small, and – denotes that the written decimal is too large; the error always being less than one half of a unit in the last place of the decimal.

Reduce to decimals:

3. $\frac{3}{4}$.	6. $\frac{14}{25}$.	9. $\frac{17}{250}$.	12. $\frac{7}{24}$.	15. $\frac{71}{82}$.
4. $\frac{5}{16}$.	7. $\frac{13}{16}$.	10. $\frac{19}{32}$.	13. $\frac{97}{160}$.	16. $\frac{5}{64}$.
5. $\frac{7}{8}$.	8. $\frac{1}{25}$.	11. $\frac{28}{12800}$.	14. $\frac{43}{56}$.	17. $\frac{35}{67}$.

Reduce to simple decimals:

18. $.24\frac{1}{2}$.	20. $.3\frac{11}{1250}$.	22. $.30\frac{1001}{148000}$.
19. $5.78\frac{19}{32}$.	21. $4.0\frac{2}{25}$.	23. $102.09\frac{3}{8}$.

284. The following fractions and decimal equivalents should be committed to memory:

$\frac{1}{2} = .50.$	$\frac{4}{5} = .80.$	$\frac{6}{7} = .85\frac{5}{7}.$	$\frac{5}{9} = .55\frac{5}{9}.$
$\frac{1}{3} = .33\frac{1}{3}.$	$\frac{1}{6} = .16\frac{2}{3}.$	$\frac{1}{8} = .12\frac{1}{2}.$	$\frac{7}{9} = .77\frac{7}{9}.$
$\frac{2}{3} = .66\frac{2}{3}.$	$\frac{5}{6} = .83\frac{1}{3}.$	$\frac{3}{8} = .37\frac{1}{2}.$	$\frac{1}{11} = .09\frac{1}{11}.$
$\frac{1}{4} = .25.$	$\frac{1}{7} = .14\frac{2}{7}.$	$\frac{5}{8} = .62\frac{1}{2}.$	$\frac{1}{12} = .08\frac{1}{3}.$
$\frac{3}{4} = .75.$	$\frac{2}{7} = .28\frac{4}{7}.$	$\frac{7}{8} = .87\frac{1}{2}.$	$\frac{1}{16} = .06\frac{1}{4}.$
$\frac{1}{5} = .20.$	$\frac{3}{7} = .42\frac{6}{7}.$	$\frac{1}{9} = .11\frac{1}{9}.$	$\frac{1}{20} = .05.$
$\frac{2}{5} = .40.$	$\frac{4}{7} = .57\frac{1}{7}.$	$\frac{2}{9} = .22\frac{2}{9}.$	$\frac{1}{25} = .04.$
$\frac{3}{5} = .60.$	$\frac{5}{7} = .71\frac{3}{7}.$	$\frac{4}{9} = .44\frac{4}{9}.$	$\frac{1}{50} = .02.$

MULTIPLICATION.

285. For multiplication of decimals by integers, see § 103. In multiplication of decimals by decimals, the location of the decimal point in the product depends upon the following principles:

PRINCIPLES. — I. *The number of ciphers in the denominator of a decimal is equal to the number of decimal places.*

II. *If we find the product of two decimals, in the fractional form, the denominator will contain as many ciphers as there are decimal places in both factors.*

III. *The product of two decimals, expressed in the decimal form, must contain as many decimal places as there are decimal places in both factors.*

Examples.

286. To multiply by a decimal.

1. Multiply .45 by .7.

OPERATION.

$$\begin{array}{r} .45 \\ .7 \\ \hline .315 \end{array}\ \textit{Ans.}$$

PROOF.

$$\tfrac{45}{100} \times \tfrac{7}{10} = \tfrac{315}{1000} = .315$$

SOLUTION. — We first multiply as in whole numbers; then, since the multiplicand has 2 decimal places and the multiplier 1, we point off 2 + 1 = 3 decimal places in the product (§ 285, III). The reason of this is further illustrated in the proof, a method applicable to all similar cases.

RULE. — *Multiply as in whole numbers, and from the right hand of the product point off as many figures for decimals as there are decimal places in both factors.*

NOTES. — 1. If there are not as many figures in the product as there are decimals in both factors, supply the deficiency by prefixing ciphers.

2. To multiply a decimal by 10, 100, 1000, etc., remove the point as many places to the right as there are ciphers on the right of the multiplier.

Multiply:

2. .75 by .41.	7. .0075 by .005.	12. 100000 by .576.
3. .436 by .24.	8. 3.24 by .324.	13. $7\frac{3}{4}$ by $5\frac{1}{2}$.
4. 5.75 by .35.	9. 75.64 by .225.	14. $.63\frac{1}{8}$ by 24.
5. .756 by .025.	10. 5.728 by 100.	15. $4\frac{5}{16}$ by $7\frac{9}{25}$.
6. 3.784 by 2.475.	11. .36 by 1000.	16. 10000 by .0001.

17. Find the value of $3.425 \times 1.265 \times 64$.

18. Find the value of $32 \times .57825 \times .25$.

19. If a cubic foot of granite weighs 168.48 pounds, what will be the weight of a granite block containing $27\frac{3}{8}$ cubic feet?

20. When a bushel of corn is worth 2.8 bushels of oats, how many bushels of oats must be given for 36.5 bushels of corn?

21. At the rate of \$9.50 on \$1000, what amount of tax should be paid on a house worth \$15575?

Contracted Multiplication.

287. It is frequently the case in multiplication of decimals, that a greater number of decimal figures is obtained in the product, than is necessary for practical accuracy. This may be avoided by contracting each partial product to the required number of decimal places.

To investigate the principles of this method, let us take the two decimals .12345 and .54321, and having reversed the order of the digits in the latter, and written it under the former, multiply each term of the direct number by the term below in the *reversed* number, placing the products with like orders of units in the same column, thus:

$$
\begin{array}{llll}
.12345 \text{ direct} & = .12345 & & \\
.54321 \text{ reversed} & = \underline{12345.} & & \\
 & .000025 & = .00005 & \times .5 \\
 & .000016 & = .0004 & \times .04 \\
 & .000009 & = .003 & \times .003 \\
 & .000004 & = .02 & \times .0002 \\
 & .000001 & = .1 & \times .00001.
\end{array}
$$

In this operation we perceive that all the products are of the same order; and this must always be the case, whether the numbers used are fractional, integral, or mixed. For, as we proceed from right to left in the multiplication, we pass regularly from lower to higher orders in the *direct* number, and from higher to lower in the *reversed* number.

PRINCIPLE.—*If one number is written under another with the order of its digits reversed, and each unit of the reversed number is multiplied by the unit above it in the direct number, the products will all be of the same order of units.*

Examples.

288. To obtain a given number of decimal places in the product.

1. Multiply 4.78567 by 3.25765, retaining only 3 decimal places in the product.

OPERATION.

$$
\begin{array}{rl}
4.78567 & \\
\underline{56752.3} & \\
14357 = & 4785 \times 3 + 2 \\
957 = & 478 \times 2 + 1 \\
239 = & 47 \times 5 + 4 \\
33 = & 4 \times 7 + 5 \\
\underline{3 =} & 0 \times 6 + 3 \\
15.589 \pm & \textit{Ans.}
\end{array}
$$

SOLUTION.—Since the product of any term by units is of the same order as the term multiplied (§ 102, II), we write 3, the units of the multiplier, under 5, the third decimal term of the multiplicand, and the lowest order to be retained in the product; and the other terms of the multiplier we write in the inverted order, extending to the left. Then, since the product of 3 and 5 is of the third order, or thousandths, the products of the other corresponding terms at the left, 2 and 8, 5 and 7, 7 and 4, etc., will be thousandths, § 287, Prin.; and we therefore multiply each term of the multiplier by the terms above and to the left of it in the multiplicand, carrying from the rejected figures of the multiplicand, as follows: 3 times 6 are 18, and as this is nearer 2 units than one of the next higher order, we must carry 2 to the first contracted product; 3 times 5 are 15, and 2 to be carried are 17; writing the 7 under the 3, and multiplying the other terms at the left in the usual manner, we obtain 14357 for the first partial product. Then, beginning with the next term of the multiplier, 2 times 5 are 10, which gives 1 to be carried to the second partial product; 2 times 8 are 16, and 1 to be carried are 17; writing the 7 under the first figure of the former product, and multiplying the remaining left-hand terms of the multiplicand, we obtain 957 for the second partial product. Then, 5 times 8 are 40, which gives 4 to be carried to the third partial product; 5 times 7 are 35 and 4 are 39; writing the 9 in the first column of the products. and proceeding as in the former steps, we obtain 239 for the third partial product. Next, multiplying by 7 in the same manner, we obtain 33 for the fourth partial product. Lastly, beginning 2 places to the right in the multiplicand, 6 times 7 are 42; 6 times 4 are 24, and 4 are 28, which gives 3 to be carried to the fifth partial product; 6 times 0 is 0, and 3 to be carried are 3, which we write for the last partial product. Adding the several partial products, and pointing off 3 decimal places, we have 15.589, the required product.

RULE.—I. *Write the multiplier with the order of its figures reversed, and with the units' term under that figure of the multiplicand which represents the lowest decimal order to be retained in the product.*

II. *Find the product of each term of the multiplier by the terms in the multiplicand above and to the left of it, increasing each partial product by as many units as would have been carried from the rejected part of the multiplicand, and one more when the highest term in the rejected part of any product is 5 or greater than 5; and write these partial products with the lowest term of each in the same column.*

III. *Add the partial products, and from the right hand of the result point off the required number of decimal places.*

NOTES. — 1. In obtaining the number to be carried to each contracted partial product, it is generally necessary to multiply (mentally) only one term at the right of the term above the multiplier; but when the terms are large, the multiplication should commence at least *two* places to the right.

2. When the number of units in the highest order of the rejected part of the product is between 5 and 15, we carry 1; when between 15 and 25, we carry 2; when between 25 and 35, we carry 3; and so on.

3. There is always a liability to an error of one or two units in the last place; and as the answer may be either too great or too small by the amount of this error, the uncertainty may be indicated by the double sign, ±, read, *plus, or minus*, and placed after the product.

4. When the number of decimal places in the multiplicand is less than the number to be retained in the product, the deficiency must be supplied by annexing ciphers.

2. Multiply 236.45 by 32.46357, retaining 2 decimal places in the product.

3. Multiply 2.563789 by .0347263, retaining 6 decimal places in the product.

2. OPERATION.	3. OPERATION.
236.450	2.563789
75364.23	3627430.
709350	76914
47290	10255
9458	1795
1419	51
71	15
12	1
2	.089031 ± *Ans.*
7676.02 ± *Ans.*	

4. Multiply 215.72 by 4.347892, retaining four decimal places in the product.

5. Multiply .24367 by 36.75, retaining 2 decimal places in the product.

6. Multiply 4256.785 by .00564, rejecting all beyond the third decimal place in the product.

7. Multiply 357.84327 by 1.007806, retaining 4 decimal places in the product.

8. Multiply 272.345 by 7.0003, retaining 4 decimal places in the product.

9. Multiply 999.999 by .0909, retaining 5 decimal places in the product.

10. Multiply 300060.5 by .04002, retaining 3 decimal places in the product.

11. Multiply 400.756 by 1.367583, retaining 2 decimal places in the product.

12. Multiply 432.5672 by 1.0666666, retaining 3 decimal places in the product.

13. Multiply 48.4367 by $2\frac{5}{87}$, extending the product to 3 decimal places.

14. Multiply $7\frac{5}{118}$ by $3\frac{376}{489}$, extending the product to 3 decimal places.

15. Multiply 36.275 by 4.3678, retaining 1 decimal place in the product.

16. Multiply 9765.493 by .0509187, retaining 5 decimal places in the product.

17. The first satellite of Uranus moves in its orbit 142.8373 + degrees in 1 day. Find how many degrees it will move in 2.52035 days, carrying the answer to 2 decimal places.

18. A gallon of distilled water weighs 8.33888 pounds. How many pounds are there in 35.8756 gallons?

19. One French meter is equal to 1.09356959 English yards. How many yards are there in 478.7862 meters?

20. The polar radius of the earth is 6356078.96 meters, and the equatorial radius, 6377397.6 meters. Find the two radii, and their difference, to the nearest hundredth of a mile, 1 meter being equal to 0.000621346 of a mile.

DIVISION.

289. For division of decimals by integers, see § 164. In division of decimals by decimals the location of the decimal point in the quotient depends upon the following principles:

PRINCIPLES. — I. *If one decimal number in the fractional form is divided by another, also in the fractional form, the denominator of the quotient must contain as many ciphers as the number of ciphers in the denominator of the dividend exceeds the number in the denominator of the divisor.*

II. *The quotient of one number divided by another in the decimal form must contain as many decimal places as the number of decimal places in the dividend exceeds the number in the divisor.*

Examples.

290. To divide by a decimal.

1. Divide 34.368 by 5.37.

OPERATION.

```
5.37) 34.368 (6.4 Ans.
      32 22
      -----
       2 148
       2 148
       -----
```

PROOF.

$$\frac{\cancel{34368}^{\,64}}{\cancel{1000}_{\,10}} \times \frac{\cancel{100}}{\cancel{537}} = \frac{64}{10} = 6.4$$

SOLUTION. — We first divide as in whole numbers; then, since the dividend has 3 decimal places and the divisor 2, we point off $3 - 2 = 1$ decimal place in the quotient (§ 285, III). The correctness of the work is shown in the proof, where the dividend and divisor are written as common fractions. For, when we have canceled the denominator of the divisor from the denominator of the dividend, the denominator of the quotient must contain as many ciphers as the number of ciphers in the dividend exceeds those in the divisor.

RULE. — *Divide as in whole numbers, and from the right hand of the quotient point off as many places for decimals as the decimal places in the dividend exceed those in the divisor.*

NOTES. — 1. If the number of figures in the quotient is less than the excess of the decimal places in the dividend over those in the divisor, the deficiency must be supplied by prefixing ciphers.

2. *If there is a remainder after* dividing the dividend, annex ciphers, and continue the *division; the ciphers annexed* are decimals of the dividend.

3. The dividend should always contain at least as many decimal places as the divisor, before commencing the division; the quotient figures will then be integers till all the decimals of the dividend have been used in the partial dividends.

4. To divide a decimal by 10, 100, 1000, etc., we must remove the point as many places to the left as there are ciphers on the right of the divisor.

Divide:

2. 96188 by 3.46.
3. 46.1975 by 54.35.
4. .014274 by .061.
5. .952 by 4.76.
6. 345.15 by .075.
7. .8 by 476.3.
8. .0026 by .003.
9. 3.6 by .00006.
10. 3 by 450.
11. 75 by 10000.
12. 4.36 by 100000.
13. 645.5 by 1000.

14. How many men will it take to build 154.125 rods of fence in 1 day if each builds 6.165 rods?

15. How many coats can be made from 16.2 yards of cloth, allowing 2.7 yards for each coat?

16. If a man travels 36.34 miles a day, how long will it take him to travel 674 miles?

17. How many revolutions will a wheel 14.25 feet in circumference make in going a distance of 1 mile or 5280 feet?

18. A man having $ 4000 wishes to exchange it for marks (German money). If 1 mark is equal to $ .238, how much will he get in exchange?

19. A city was taxed $ 575 on $ 650000 worth of property. What was the tax on a dollar, and what was the amount of Mr. Jefferson's tax, who owned $ 569 worth of taxable property?

20. A man having $ 973.30 wishes to exchange it for English money. At $ 4.8665 to the pound, how many pounds sterling would he receive in exchange?

21. What is the diameter of a circle whose circumference is 35 feet, the circumference of a circle being 3.1416 times its diameter?

22. The product of four numbers is 932.25; three of them are 56.5, 1.1, and .03. What is the fourth?

CONTRACTED DIVISION.

291. In division of decimals, the products of the divisor by the several quotient terms may be contracted, as in multiplication, by rejecting at each step the unnecessary figures of the divisor (§ 287).

Examples.

292. To obtain a given number of decimal places in the quotient.

1. Divide 790.755197 by 32.4687, extending the quotient to 2 decimal places.

FIRST CONTRACTED METHOD.

```
32.4687 ) 790.755197 (24.35
          649 4               Ans.
          -----
          141 3
          129 9
          -----
           11 4
            9 7
           ----
            1 7
            1 6
           ----
              1
```

COMMON METHOD.

```
32.4687 ) 790.7|55198 (24.35
          649 3|74           Ans.
          -----|--
          141 3|811
          129 8|748
          -----|---
           11 5|0639
            9 7|4061
           ----|----
            1 7|65787
            1 6|23435
           ----|-----
              1|42352
```

SECOND CONTRACTED METHOD.

```
32.4687 ) 790.755197
53.42       141 3
             11 4
              1 7
                1
```

SOLUTION. — In the first method of contraction, we first compare the 3 tens of the divisor with the 79 tens of the dividend, and ascertain that there will be 2 integral places in the quotient; and as 2 decimal places are required, the quotient must contain 4 places in all. Then assuming the four left-hand terms of the divisor, we say 3246 is contained in 7907, 2 times; multiplying the assumed part of the divisor by 2, and carrying 2 units from the rejected part, as in Contracted Multiplication of Decimals, we have 6494 for the product, which, subtracted from the dividend, leaves 1413 for a new dividend. Now, since the next quotient term will be of an order next below the former, we reject one more place in the divisor, and divide by 324, obtaining 4 for a quotient, 1299 for a product, and 114 for a new dividend. Continuing this process till all the terms of the divisor are rejected, we have, after pointing off 2 decimals as required, 24.35 for a quotient. Com-

paring the contracted method with the common method, we see the extent of the abbreviation, and the agreement of the corresponding intermediate results.

In the second method of contraction, the quotient is written with its first term under the lowest order of the assumed divisor, and the other terms at the left in the reverse order. By this arrangement, the several products are conveniently formed, by multiplying each quotient term by the terms above and to the left of it in the divisor, by the rule for contracted multiplication (§ 288), and the *remainders* only are written as in § 173.

RULE. — I. *Compare the highest or left-hand term of the divisor with the units of like order in the dividend, and determine how many terms will be required in the quotient.*

II. *For the first contracted divisor take as many significant terms from the left of the given divisor as there are places required in the quotient; and at each subsequent division reject one place from the right of the last preceding divisor.*

III. *In multiplying by the several quotient terms carry from the rejected terms of the divisor as in contracted multiplication.*

NOTES. — 1. Supply ciphers, at the right of either divisor or dividend, when necessary, before commencing the work.

2. If the first term obtained in the quotient is written under the lowest assumed term of the divisor, and the others at the left in the inverted order, the several products will be formed with the greatest convenience, by simply multiplying each quotient term by the term above and to the left of it in the divisor.

2. Divide 27.3782 by 4.3267, extending the quotient to 3 decimal places.

3. Divide 487.24 by 1.003675, extending the quotient to 2 decimal places.

4. Divide 8.47326 by 75.43, extending the quotient to 5 decimal places.

5. Divide .8487564 by .075637, extending the quotient to 3 decimal places.

6. Divide 478.325 by 1.43⅜, extending the quotient to 3 decimal places.

7. Divide 8972.436 by 756.3452, extending the quotient to 4 decimal places.

8. Divide 1 by 1.007633, extending the quotient to 6 decimal places.

CIRCULATING DECIMALS.

293. Common fractions cannot always be exactly expressed in the decimal form; for in some instances the division will not be exact if continued indefinitely.

294. A **Finite Decimal** is a decimal which terminates at a certain decimal place; as .5, .75, .375.

295. An **Infinite** or **Circulating Decimal** is one which never terminates, but has one or more figures constantly repeated; as .333+, .437437+.

Circulating Decimals are also called **Circulates** or **Circulators.**

296. A **Repetend** or a recurring period is the figure or set of figures continually repeated. When a repetend consists of a single figure, it is indicated by a point placed over it; when it consists of more than one figure, a point is placed over the first, and one over the last figure. Thus, the circulating decimals .55555+ and .324324324+, are written, $.\dot{5}$ and $.\dot{3}2\dot{4}$.

297. A repetend is said to be *expanded* when its figures are continued in their proper order any number of places toward the right; thus, $.\dot{2}\dot{4}$, expanded, is .2424+, or .242424242+.

298. Repetends which begin at the same place are **Coöriginous**; those which end at the same place are **Conterminous**, and those which begin and end at the place are **Similar.**

Thus, $.2\dot{4}$ and $.3\dot{7}\dot{5}$ are coöriginous, $.7\dot{5}$ and $1.5\dot{3}$ are conterminous; $.4\dot{2}\dot{7}$ and $.5\dot{3}\dot{6}$ are similar.

NOTE.—Some authorities use the word *similar* to designate repetends which *begin* at the same place, and *like* to designate those which *begin* and *end* at the same place; but the classification into coöriginous, conterminous, and similar, as defined above, is more logical.

299. **A Pure Circulating Decimal** is one which contains no figures but the repetend; as $.\dot{7}$, or $.\dot{7}0\dot{4}$.

300. **A Mixed Circulating Decimal** is one which contains other figures, called *finite places*, before the repetend; as $.5\dot{4}$, or $.01\dot{3}24\dot{5}$, in which .5 and .01 are the finite places.

PROPERTIES OF FINITE AND CIRCULATING DECIMALS.

301. The operations in circulating decimals depend upon the following principles:

NOTE.—The common fractions referred to are understood to be *proper* fractions, in their *lowest terms*.

PRINCIPLES.—I. *Every fraction in its lowest terms whose denominator contains no other prime factor than 2 or 5 will give rise to a finite decimal; and the number of decimal places will be equal to the greatest number of equal factors, 2 or 5, in the denominator.*

For, in the reduction, every cipher annexed to the numerator multiplies it by 10, or introduces the two prime factors, 2 and 5, and also gives 1 decimal place in the result. Hence the division will be exact when the number of ciphers annexed, or the number of decimal places obtained, shall be equal to the greatest number of equal factors, 2 or 5, to be canceled from the denominator.

II. *Every fraction whose denominator contains any other prime factor than 2 or 5, will give rise to an infinite or circulating decimal.*

For, annexing ciphers to the numerator introduces no other prime factors than 2 and 5; hence the numerator will never contain all the prime factors of the denominator.

III. *The number of places in the repetend of a circulating decimal must be less than the number of units in the denominator of the common fraction.*

For, in every division, the number of possible remainders is limited to the number of units in the divisor, less 1; thus, in dividing by 7, the only possible remainders are 1, 2, 3, 4, 5, and 6. Hence, in the reduction of a common fraction to a decimal, some of the remainders must repeat before the number of decimal places obtained equals the number of units in the denominator; and this will cause the intermediate quotient figures to repeat.

NOTE. — 1. It will be found that the number of places in the repetend is always equal to the denominator less 1, or to some factor of this number. Thus, the repetend arising from $\frac{1}{7}$ has $7-1=6$ places; the repetend arising from $\frac{1}{41}$ has $\frac{41-1}{8}=5$ places.

2. A *perfect repetend* is one which consists of as many places, less 1, as there are units in the denominator of the equivalent fraction.

3. If the denominator of a fraction contains neither of the factors 2 and 5, it will give rise to a pure repetend. But if a circulating decimal is derived from a fraction whose denominator contains either of the factors 2 or 5, it will contain as many finite places as the greatest number of equal factors 2 or 5 in the denominator.

IV. *If to any number we annex as many ciphers as there are places in the number, or more, and divide the result by as many 9's as the number of ciphers annexed, both the quotient and remainder will be the same as the given number.*

For, if we take any number of two places, as 74, and annex two ciphers, the result divided by 100 will be equal to 74; thus, $7400 \div 100 = 74$. Now subtracting 1 from the divisor, 100, will add as many units to the quotient, 74, as the new divisor, 99, is contained times in 74 (§ 167, II); thus, $7400 \div 99 = 74 + \frac{74}{99}$, or $74\frac{74}{99}$; that is, if two ciphers are annexed to 74, and the result is divided by 99, both quotient and remainder will be 74. In like manner, annexing three ciphers to 74, and dividing by 999, we have $74000 \div 999 = 74\frac{74}{999}$; and the same is true of any number whatever.

V. *Every pure circulating decimal is equal to a common fraction whose numerator is the repeating figure or figures, and whose denominator is as many 9's as there are places in the repetend.*

For, if we take any fraction whose denominator is expressed by some number of 9's, as $\frac{24}{99}$, then according to the last property, annexing two ciphers to the numerator, and reducing to a decimal, we have $\frac{24}{99} = .24\frac{24}{99}$. In like manner, carrying the decimal two places farther, $.24\frac{24}{99} = .2424\frac{24}{99}$; hence, $\frac{24}{99} = .\dot{2}\dot{4}$. By the same principle, we have $\frac{2}{9} = .\dot{2}$; $\frac{1}{99} = .\dot{0}\dot{1}$; $\frac{2}{99} = .\dot{0}\dot{2}$; $\frac{1}{999} = .\dot{0}0\dot{1}$; $\frac{324}{999} = .\dot{3}2\dot{4}$; and so on. And it is evident that *all possible repetends* can thus be derived from fractions whose numerators are the repeating figures, and whose denominators are as many 9's as there are repeating figures.

NOTE. — It follows from the last property, that any fraction from which a pure repetend can be derived is reducible to a form in which the denominator is some number of 9's; thus $\frac{2}{13} = \frac{153846}{999999}$; $\frac{5}{37} = \frac{135}{999}$. This is true of every fraction whose denominator terminates with 1, 3, 7, or 9.

VI. *A common fraction whose denominator contains 2's or 5's with other prime factors will give a mixed circulating decimal and the number of places in the non-repeating or finite part will equal the greatest number of 2's or 5's in the denominator.*

For dividing first by the 2's and 5's, we shall have a decimal numerator containing as many places as the greatest number of 2's or 5's (I). Dividing by any other factor will give rise to the repetend (II). Hence the two divisions will produce a mixed circulating decimal. $\frac{1}{70} = \frac{1}{2 \times 5 \times 7}$. Dividing first by 2×5, or 10, we have $\frac{.1}{7}$. Dividing .1 by 7, we have the mixed circulate $.0\dot{1}4285\dot{7}$, which contains *one* finite place.

VII. *The number of repetend places will not exceed the number expressed by the product of all the prime factors of the denominator other than* 2 *or* 5.

For the factors 2 or 5 give rise to the finite part of the decimal and the other factors to the repetend.

VIII. *Any repetend may be reduced to another equivalent repetend, by expanding it, and moving either the second point, or both points, to the right; provided that in the result they are so placed as to include the same number of places as are contained in the given repetend, or some multiple of this number.*

For, in every such reduction, the new repetend and the given repetend, when expanded indefinitely, will give results which are identical. *Thus*, $.\dot{5}3\dot{6} = .\dot{5}3653\dot{6}$, or $.\dot{5}3653653\dot{6}$, or $.5\dot{3}6\dot{5}$, or $.53\dot{6}5\dot{3}$, or $.5\dot{3}6536\dot{5}$, or $.53\dot{6}5365365\dot{3}$; because each of these new repetends, when expanded, gives $.536536536536536536536+$.

NOTE. — If in any reduction, the new repetend should not contain the same number of places, or some multiple of the same number, as the given repetend, we should not have, in the expansions, the *same figures* repeated in the *same order*.

REDUCTION.

Examples.

302. To reduce a pure circulating decimal to a common fraction.

1. Reduce $.\dot{6}7\dot{5}$ to a common fraction.

OPERATION.

$.\dot{6}7\dot{5} = \frac{675}{999} = \frac{25}{37}$ *Ans.*

SOLUTION. — Since the repetend has 3 places, we take for the denominator of the required fraction the number expressed by three 9's (§ 301, V).

RULE. — *Omit the points and the decimal sign, and write the figures of the repetend for the numerator of a common fraction, and as many 9's as there are places in the repetend for the de-*

Reduce to common fractions or mixed numbers:

2. $.\dot{4}\dot{5}$.	4. $.\dot{2}7\dot{9}$.	6. $.\dot{9}2307\dot{6}$.	8. $4.\dot{7}\dot{2}$.	10. $\dot{2}.9\dot{7}$.
3. $.\dot{6}\dot{6}$.	5. $.\dot{4}2\dot{3}$.	7. $.\dot{9}512\dot{1}$.	9. $2.\dot{2}9\dot{7}$.	11. $\dot{1}5.\dot{0}$.

NOTE. — According to § 301, VIII, $\dot{2}.9\dot{7} = 2.\dot{9}7\dot{2}$.

303. To reduce a mixed circulating decimal to a common fraction.

1. Reduce $.0\dot{7}5\dot{6}$ to a common fraction.

OPERATION.

$.0\dot{7}5\dot{6} = \frac{756}{9990} = \frac{14}{185}$ *Ans.*

SOLUTION. — Since $.\dot{7}5\dot{6}$ is equal to $\frac{756}{999}$, $.0\dot{7}5\dot{6}$ will be $\frac{1}{10}$ of $\frac{756}{999}$, or $\frac{756}{9990} = \frac{14}{185}$.

2. Reduce $.64\dot{7}$ to a common fraction.

OPERATION.

$$.64\dot{7} = \frac{64}{100} + \frac{7}{900} = \frac{640 - 64}{900} + \frac{7}{900} = \frac{640 - 64 + 7}{900} = \frac{647 - 64}{900} = \frac{583}{900} \text{ Ans.}$$

Or,

$.64\dot{7}$ given decimal.
64 finite figures.
583; $\frac{583}{900}$ *Ans.*

SOLUTION. — Reducing the finite part and the repetend separately to fractions, we have $\frac{64}{100} + \frac{7}{900}$. To reduce these fractions to a common denominator, we must multiply the terms of the first by 9; but the numerator, 64, may be multiplied by 9 by annexing 1 cipher and subtracting 64 from the result, giving $\frac{640 - 64}{900}$, for the first fraction reduced. The numerator of the sum of the two fractions will therefore be $640 - 64 + 7 = 583$, and supplying the common denominator, we have $\frac{583}{900}$. In the second operation, the intermediate steps are omitted.

RULE. — I. *From the given circulating decimal subtract the finite part, and the remainder will be the required numerator.*

II. *Write as many 9's as there are repetend places, with as many ciphers annexed as there are finite places, for the required denominator.*

Reduce to common fractions:

3. $.5\dot{7}$.	5. $.64\dot{7}\dot{2}$.	7. $.04\dot{6}4\dot{8}$.	9. $.9\dot{2}8571\dot{4}$.	11. $7.0\dot{1}2\dot{6}$.
4. $.04\dot{8}$.	6. $.65\dot{9}\dot{0}$.	8. $.100\dot{4}$.	10. $5.2\dot{7}$.	12. $2.0\dot{2}926\dot{8}$.

304. To make two or more repetends similar.

1. Make $.\dot{4}\dot{7}$, $.53\dot{6}7\dot{5}$, and $.3\dot{7}23\dot{4}$ similar.

OPERATION.

$$\left.\begin{array}{ll} .\dot{4}\dot{7} & = .47\dot{4}74747474747\dot{7} \\ .53\dot{6}7\dot{5} & = .53\dot{6}7567567567\dot{5} \\ .3\dot{7}23\dot{4} & = .37\dot{2}34723472347\dot{7} \end{array}\right\} Ans.$$

SOLUTION. — The first of the given repetends begins at the place of tenths, the second at the place of thousandths, and the third at the place of hundredths; and as the points in any repetend cannot be moved to the left over the finite places, we can make the given repetends *similar*, only by moving the points of at least two of them to the right.

Again, the first repetend has 2 places, the second 3 places, and the third 4 places; and the number of places in the new repetends must be at least 12, which is the least common multiple of 2, 3, and 4. We therefore expand the given repetends, and place the first point in each new repetend over the third place in the decimal, and the second point over the fourteenth, and thus render them similar.

RULE. — I. *Expand the repetends, and place the first point in each over the same order in the decimal.*

II. *Place the second point so that each new repetend shall contain as many places as there are units in the least common multiple of the number of places in the several given repetends.*

NOTES. — 1. Since none of the points can be carried to the left, some of them must be carried to the right, so that each repetend shall have at least as many finite places as the greatest number in any of the given repetends.

2. Any finite decimal may be considered as a circulating decimal whose repetend is 0, and can be made similar to a circulating decimal by § 304.

2. Make $.4\dot{3}$, $.\dot{5}\dot{7}$, $.4\dot{5}6\dot{7}$, and $.50\dot{3}\dot{7}$ similar.
3. Make $.57\dot{8}$, $.\dot{3}\dot{7}$, $.248\dot{5}$, and $.\dot{0}\dot{4}$ similar.
4. Make $\dot{1}.3\dot{4}$, $\dot{4}.5\dot{6}$, and $.3\dot{4}\dot{1}$ similar.
5. Make $.\dot{5}67\dot{4}$, $.3\dot{4}$, $.\dot{2}4\dot{7}$, and $.\dot{6}\dot{7}$ similar.
6. Make $\dot{1}.2\dot{4}$, $.0\dot{5}7\dot{8}$, $.\dot{4}$, and $.47\dot{3}214\dot{7}$ similar.
7. Make $.\dot{7}$, $.456\dot{7}$, $.\dot{2}\dot{4}$, and $.3\dot{4}678\dot{9}$ similar.
8. Make $.\dot{8}$, $.\dot{3}\dot{6}$, $.\dot{4}85\dot{7}$, $.\dot{3}456\dot{7}$, and $.\dot{2}78467894\dot{3}$ similar.
9. Make $.57\dot{8}$, $.3\dot{4}\dot{1}$, $.\dot{4}85\dot{7}$, $.456\dot{7}$, and $.47\dot{3}214\dot{7}$ similar.
10. Make $.248\dot{5}$, $.\dot{4}\dot{5}$, $.3\dot{4}1$, and $\dot{1}.3\dot{4}$ similar.

ADDITION AND SUBTRACTION.

305. The processes of adding and subtracting circulating decimals depend upon the following properties of repetends:

PRINCIPLES. — I. *If two or more repetends are similar, their denominators will consist of the same number of 9's, with the same number of ciphers annexed.*

II. *Similar repetends have the same denominators and consequently the same fractional unit.*

Examples.

306. To add and subtract circulating decimals.

1. Add $.5\dot{4}$, $3.\dot{2}\dot{4}$, and $\dot{2}.78\dot{5}$.

OPERATION.

$$\begin{aligned} .5\dot{4} &= .5\dot{4}44\dot{4} \\ 3.\dot{2}\dot{4} &= 3.2\dot{4}24\dot{2} \\ \dot{2}.78\dot{5} &= 2.7\dot{8}52\dot{7} \\ \hline & \ 6.5\dot{7}21\dot{4}\ \textit{Ans.} \end{aligned}$$

SOLUTION. — Since fractions can be added only when they have the same fractional unit, we first make the repetends of the given decimals similar. We then add as in finite decimals, observing, however, that the 1 which we carry from the left-hand column of the repetends must also be added to the right-hand column; for this would be required if the repetends were further expanded before adding.

2. From $7.\dot{4}$ take $2.7\dot{8}5\dot{2}$.

OPERATION.

$$\begin{array}{l} 7.4\dot{4}4\dot{4} \\ \underline{2.7\dot{8}5\dot{2}} \\ 4.6\dot{5}8\dot{1}\ \textit{Ans.} \end{array}$$

SOLUTION. —Since one fraction can be subtracted from another only when they have the same fractional unit, we first make the repetends of the given decimals similar. We then subtract as in finite decimals; observing that if both repetends were expanded, the next figure in the subtrahend would be 8, and the next in the minuend 4; and the subtraction in this form would require 1 to be carried to the 2, giving 1 for the right-hand figure in the remainder.

RULE. — I. *When necessary, make the repetends similar.*

II. To add. —*Proceed as in finite decimals, and remember to increase the sum of the right-hand column by as many units as are carried from the left-hand column of the repetends.*

III. To subtract. —*Proceed as in finite decimals and diminish the right-hand figure of the remainder by 1, when the repetend in the subtrahend is greater than the repetend in the minuend.*

IV. *Place the points in the result directly under those above.*

NOTE. — When the sum or difference is required in the form of a common fraction, proceed according to the rule, and reduce the result.

Add:

3. $.\dot{5}$, $.\dot{3}\dot{2}$, and $.\dot{1}\dot{2}$.
4. $.4\dot{3}\dot{8}7$, $.863$, $.2\dot{1}$ and $.3\dot{5}5\dot{4}$.
5. $2.\dot{4}$, $.32$, $.\dot{5}6\dot{7}$, and $7.0\dot{5}\dot{6}$.
6. $4.6\dot{3}\dot{8}$, $8.3\dot{1}\dot{8}$, $.\dot{0}16$ and $.\dot{4}\dot{5}$.
7. From $.43\dot{2}$ take $.25$.
8. From $7.2457\dot{4}$ take $2.\dot{6}3\dot{4}$.
9. From $.99$ take $.\dot{4}3\dot{3}$.
10. From $.\dot{4}$ take $.\dot{2}\dot{3}$.

MULTIPLICATION AND DIVISION.

Examples.

307. To multiply and divide circulating decimals.

1. Multiply $2.\dot{4}28571\dot{}$ by $.0\dot{6}\dot{3}$.

OPERATION.

$$2.\dot{4}2857\dot{1} = 2\tfrac{428571}{999999} = 2\tfrac{3}{7} = \tfrac{17}{7}$$
$$.0\dot{6}\dot{3} = \tfrac{63}{990} = \tfrac{7}{110}$$
$$\frac{17}{\cancel{7}} \times \frac{\cancel{7}}{110} = \frac{17}{110} = .1\dot{5}\dot{4}\ \textit{Ans.}$$

SOLUTION. — We first reduce the multiplicand and multiplier to their equivalent fractions and obtain $\frac{17}{7}$ and $\frac{7}{110}$; then $\frac{17}{7} \times \frac{7}{110} = \frac{17}{110} = .1\dot{5}\dot{4}$.

2. Divide $.\dot{4}7\dot{5}$ by $3\dot{7}5\dot{0}$.

OPERATION.

$$.\dot{4}7\dot{5} = \tfrac{475}{999}$$
$$.3\dot{7}5\dot{0} = \tfrac{3747}{9990}$$
$$\tfrac{475}{999} \times \tfrac{9990}{3747} = 1.26\ \textit{Ans.}$$

SOLUTION. — The dividend reduced to its equivalent common fraction is $\frac{475}{999}$, and the divisor reduced to its equivalent common fraction is $\frac{3747}{9990}$; and $\frac{475}{999} \div \frac{3747}{9990} = \frac{19}{15} = 1.2\dot{6}$.

RULE. — *Reduce the given numbers to common fractions; then multiply or divide, and reduce the result to a decimal.*

3. Multiply 3.4 by $.7\dot{2}$.
4. Multiply $.0\dot{4}3\dot{2}$ by 18.
5. Divide $.\dot{1}5\dot{4}$ by $.\dot{2}$.
6. Divide $4.572\dot{4}$ by $.\dot{7}$.
7. Multiply 4.37 by $.27$.
8. Divide $56.\dot{6}$ by 137.
9. Divide $.\dot{4}2857\dot{1}$ by $.\dot{5}\dot{4}$.
10. Multiply $.\dot{7}1428\dot{5}$ by $.\dot{2}\dot{7}$.
11. Multiply $3.4\dot{5}\dot{6}$ by $.425$.
12. Divide $9.\dot{1}704\dot{5}$ by $3.\dot{3}\dot{6}$.
13. Multiply $.0\dot{5}7\dot{8}$ by $.\dot{4}$.
14. Divide $.\dot{4}85\dot{7}$ by $.3\dot{7}$.

MEASURES — METRIC AND COMMON.

308. **Measure** is that by which extent, dimension, capacity, quantity of matter, or money value is ascertained, determined according to some fixed standard.

NOTE. — The process by which the extent, dimension, capacity, etc., is ascertained, is called *measuring*; and consists in comparing the thing to be measured with some conventional standard or *unit of measure*.

309. Measures are of seven kinds:

1. Length.
2. Surface or Area.
3. Volume or Capacity.
4. Weight, or Force of Gravity.
5. Time.
6. Angles.
7. Money or Value.

The first three kinds may be properly divided into two classes, — Measures of Extension, and Measures of Capacity.

310. A **Table** is a regular arrangement of the denominations used to express any measure, stating the number of units of each denomination equal to a unit of the next higher denomination.

GOVERNMENT STANDARDS OF MEASURES.

311. The English, American, and French Governments, in establishing their standards of measures and weights, founded them upon unalterable principles or laws of nature, as will be seen by examining the several standards.

NOTE. — In early times, almost every province and chief city had its own measures and weights; but these were neither definite nor uniform. This variety in the weights and measures of different countries has always proved a serious embarrassment to commerce; hence the many attempts that have been made in modern times to establish uniformity.

UNITED STATES STANDARDS.

312. In the year 1834 the United States Government adopted a uniform standard of weights and measures, for the use of the customhouses and the other branches of business connected with the general government. Most of the states which have adopted any standards have taken those of the general government. The *invariable standard unit* from which all other standard units of measure are derived is the **Day.**

Astronomers have proved that the diurnal revolution of the earth is *entirely uniform*, always performing equal parts of a revolution on its axis in equal periods of duration.

Having decided upon the invariable standard unit, a measure of this unit was sought that should in some manner be connected with extension as well as with this unit. A clock pendulum whose rod is of any given length, is found always to vibrate the same number of times in the same period of duration. From the day and the pendulum, the different standards hereafter given were determined and adopted.

United States Standard of Extension.

313. The *United States standard unit of measures of extension*, whether linear, superficial, or solid, is the yard of 3 feet, or 36 inches, and is the same as the Imperial standard yard of Great Britain.

It is determined as follows: The rod of a pendulum vibrating seconds of mean time, in the latitude of London, in a vacuum, at the level of the sea, is divided into 391393 equal parts, and 360000 of these parts are 36 inches, or 1 standard yard. Hence, such a pendulum rod is 39.1393 inches long, and the standard yard is $\frac{360000}{391393}$ of the length of the pendulum rod.

NOTE. — 1. It is impracticable to reproduce the yard from the pendulum, and the practical standard is a metal rod at Washington, from which duplicates are furnished.

2. The old British Bird's standard yard of 1760 was found inadequate, and a new standard was constructed. This was destroyed by fire in the burning of the houses of Parliament in 1834, and a new standard, reproduced by reference to all the former standards, was constructed in 1844. This was known as the British Imperial yard.

3. The British Imperial yard is defined by Act of Parliament as the distance between the centers of two cylindrical holes in a certain bar of bronze when the metal has a temperature of 62° Fahrenheit.

United States Standards of Capacity.

314. The *United States standard unit of liquid measure* is the old English wine gallon, of 231 cubic inches, which is equal to 8.33888 pounds avoirdupois of distilled water at its maximum density; that is, at the temperature of 39.83° Fahrenheit, the barometer being at 30 inches.

315. The *United States standard unit of dry measure* is the British Winchester bushel, which is $18\frac{1}{2}$ inches in diameter and 8 inches deep, and contains 2150.42 cubic inches, equal to 77.6274 pounds avoirdupois of distilled water, at its maximum density. A gallon, dry measure, contains 268.8 cubic inches.

United States Standards of Weight.

316. It has been found that a given volume or quantity of distilled rain water at a given temperature always weighs the same. Hence, a cubic inch of distilled rain water has been adopted as the standard of weight.

317. The *United States standard unit of weight* is the Troy pound of the Mint, which is the same as the Imperial standard pound of Great Britain, and is determined as follows: A cubic inch of distilled water in a vacuum, weighed by brass weights, also in a vacuum, at a temperature of 62° Fahrenheit, is equal to 252.458 grains, of which the standard Troy pound contains 5760.

318. The *United States Avoirdupois pound* is determined from the standard Troy pound, and contains 7000 Troy grains. Hence, the Troy pound is $\frac{5760}{7000} = \frac{144}{175}$ of an avoirdupois pound. But the Troy ounce contains $\frac{5760}{12} = 480$ grains, and the avoirdupois ounce $\frac{7000}{16} = 437.5$ grains; and an ounce Troy is $480 - 437.5 = 42.5$ grains greater than an ounce avoirdupois. The pound, ounce, and grain, apothecaries' weight, are the same as the like denominations in Troy weight, the only difference in the two tables being in the divisions of the ounce.

NITED STATES STANDARD SETS OF WEIGHTS AND MEASURES.

319. A uniform set of weights and measures for all the states was approved by Congress, June 14, 1836, and furnished to the states in 1842. The set furnished consisted of:

1. A yard.
2. A set of Troy weights.
3. A set of avoirdupois weights.
4. A wine gallon and its subdivisions.
5. A half bushel and its subdivisions.

320. Each state furnishes standard sets of weights and measures to its counties and towns. A **County Standard** may consist of:

1. A large balance, comprising a brass beam and scale dishes, with stand and lever.
2. A small balance, with a drawer stand for small weights.
3. A set of large brass weights, namely, 50, 25, 20, 10, and 5 lb.
4. A set of small brass weights, avoirdupois, namely, 4, 2, and 1 lb., 8, 4, 2, 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, and $\frac{1}{16}$ oz.
5. A brass yard measure, graduated to feet and inches, and the first foot graduated to eighths of an inch, and also decimally; with a graduation to cloth measure on the opposite side; in a case.
6. A set of liquid measures, made of copper, namely, 1 gal., $\frac{1}{2}$ gal., 1 qt., 1 pt., $\frac{1}{2}$ pt., 1 gi.; in a case.
7. A set of dry measures, of copper, namely $\frac{1}{2}$ bu., 1 pk., $\frac{1}{2}$ pk. (or 1 gal.), 2 qt. (or $\frac{1}{2}$ gal.), 1 qt.; in a case.

ENGLISH STANDARDS.

321. The English act establishing standard measures and weights, called "The Act of Uniformity," took effect Jan. 1, 1826, and the standards then adopted form what is called the **Imperial System**.

322. The *invariable standard unit* of this system is the same as that of the United States, and is described in the Act of Uniformity as follows: "Take a pendulum which will vibrate seconds in London, on a level of the sea, in a vacuum; divide all that part thereof which lies between the axis of suspension and the center of oscillation, into 391393 equal parts; then will 10000 of those parts be an imperial inch, 12 whereof make a foot, and 36 whereof make a yard."

English Standard of Extension.

323. The *English standard unit of measures of extension*, whether linear, superficial, or solid, is identical with that of the United States (§ 313).

English Standards of Capacity.

324. The *Imperial standard gallon* for liquids and all dry substances, is a measure that will contain 10 pounds avoirdupois weight of distilled water, weighed in air, at 62° Fahrenheit, the barometer at 30 inches. It contains 277.274 cubic inches.

325. The *Imperial standard bushel* is equal to 8 gallons or 80 pounds of distilled water, weighed in the manner above described. It contains 2218.192 cubic inches.

English Standards of Weight.

326. The *Imperial standard pound* is the pound Troy, which is identical with that of the United States Standard Troy pound of the mint (§ 317).

327. The *Imperial avoirdupois pound* contains 7000 Troy grains, and the Troy pound 5760 grains. They are identical with the United States avoirdupois and Troy pounds.

ENGLISH MEASURES.

328. The denominations in the standard tables of measures of extension, capacity, and weights, are the same in Great Britain and the United States. But some denominations in several of the tables are in use in various parts of Great Britain that are not known in the United States.

These denominations are retained in use by common consent, and are recognized by the English common law. They are as follows:

329. MEASURES OF EXTENSION.

18 inches	= 1 cubit.
45 inches or 5 quarters of the standard yard	= 1 ell.

NOTE. — The cubit was originally the length of a man's forearm and hand; or the distance from the elbow to the end of the middle finger.

330. MEASURES OF CAPACITY.

LIQUID MEASURE.

9 old ale gallons	= 1 firkin.
4 firkins	= 1 barrel of beer.
$7\frac{1}{2}$ Imperial "	= 1 firkin.
$52\frac{1}{2}$ Imperial gallons or 63 wine "	= 1 hogshead.
70 Imperial gallons or 84 wine "	= 1 puncheon or = $\frac{1}{3}$ of a tun.
2 hogsheads, that is 105 Imperial gallons or 126 wine "	= 1 pipe.
2 pipes	= 1 tun.

Pipes of wine are of different capacities, as follows:

110 wine gallons	= 1 pipe of	Madeira.
120 " "	= 1 "	Barcelona, Vidonia, or Teneriffe.
130 " "	= 1 "	Sherry.
138 " "	= 1 "	Port.
140 " "	= 1 "	Bucellas, or Lisbon.

331. DRY MEASURE.

8 bushels of 70 pounds each	= 1 quarter of wheat.
36 " heaped measure,	= 1 chaldron of coal.

NOTE. — The quarter of wheat is 560 pounds, or $\frac{1}{4}$ of a ton of 2240 pounds.

332. WEIGHTS.

8 pounds of butchers' meat	= 1 stone.
14 " " other commodities	= 1 " or $\frac{1}{8}$ of a cwt.
2 stone, or 28 pounds	= 1 todd of wool.
70 pounds of salt	= 1 bushel.

NOTE. — The English quarter is 28 pounds, the hundredweight is 112 pounds, and the ton is 20 hundredweight, or 2240 pounds.

FRENCH MEASURES. — THE METRIC SYSTEM.

333. The tables of standard measures and weights adopted by the French Government constitute what is called the **French Metric System.**

NOTE. — The Metric System was adopted in France in 1795; its use was authorized in Great Britain in 1864; and in 1866, Congress authorized it to be used in the United States. It is in general use by scientific men throughout the world.

334. The Metric System of weights and measures is based upon the **Decimal Scale.**

The **Meter** is the *base* of the system, and is the *one ten-millionth* part of the distance on the earth's surface from the equator to either pole, or 39.37079 inches. From the meter are made the following units: land or square measure, **Are** and **Square Meter**; wood or cubic measure, **Stere** and **Cubic Meter**; capacity, **Liter**; weight, **Gram.** These constitute the *primary* units of the system, from which all the others are derived.

The **Multiple Units,** or higher denominations, are named by prefixing to the name of the *primary* units the Greek numerals, *deca* (10), *hecto* (100), *kilo* (1000), and *myria* (10000). The **Submultiple Units,** or lower denominations, are named by prefixing the Latin numerals, *deci* ($\frac{1}{10}$), *centi* ($\frac{1}{100}$), *milli* ($\frac{1}{1000}$). By adding these prefixes to the standard units all the metrical tables are formed.

Hence, it is apparent from the *name* of a unit whether it is *greater* or *less* than the standard unit, and also *how many times.*

335. The following comparison of our ordinary decimal notation and the metric notation shows them to be the same:

Ten Thousands.	Thousands.	Hundreds.	Tens.	UNITS.		tenths.	hundredths.	thousandths.
1	1	1	1	1	.	1	1	1
Myrias	Kilos	Hectos	Decas	UNITS		decis	centis	millis

NOTES. — 1. The principal point of superiority of the metric tables is their decimal scale. By means of the decimal point several denominations may be written together as one number, as in ordinary notation, and a change to higher or lower denominations is effected by simply moving the decimal point to the left or right.

2. One disadvantage in the use of the metric system lies in the fact that with us the other denominations are in such common use that it would require a careful study and comparison of the two systems, on the part of the people at large, to make the use of the metric system practicable. Again, some of the units, while well adapted to the social and industrial conditions of France and other European countries, would not be convenient for use here. For instance, the *are*, the unit of land measure, is $\frac{1}{40}$ of an acre. While this suits the peasant farms of France very well, it would be absurdly small for a farm in the United States.

3. The metric system is now used in France, Netherlands, Spain, Italy, Greece, Austria, Germany, Norway, Sweden, Switzerland, Portugal, Mexico, Brazil, Venezuela, Argentine Republic, Haiti, and other states; and to some extent in Great Britain, the United States, India, Canada, and Chile.

UNITED STATES MEASURES.

Measures of Extension.

336. Extension has three dimensions, — length, breadth, and thickness.

A Line has only one dimension, — *length.*

A Surface or **Area** has two dimensions, — *length and breadth.*

A Solid or **Body** has three dimensions, — *length, breadth, and thickness.*

LONG MEASURE.

337. Long Measure, or **Linear Measure,** is used in measuring lines or distances. The *unit* of linear measure is the **Yard,** and the table is made up of the divisors (feet and inches), and the multiples (rods, furlongs, and miles), of this unit.

TABLE.

12	inches (in.)	= 1 foot (ft.).
3	feet	= 1 yard (yd.).
5½	yards, or 16½ feet	= 1 rod (rd.).
40	rods	= 1 furlong (fur.).
8	furlongs, or 320 rods	= 1 statute mile (mi.).

UNIT EQUIVALENTS.

mi.		fur.		rd.		yd.		ft.		in.
								1	=	12
						1	=	3	=	36
				1	=	5½	=	16½	=	198
		1	=	40	=	220	=	660	=	7920
1	=	8	=	320	=	1760	=	5280	=	63360

Scale — ascending, 12, 3, 5½, 40, 8; descending, 8, 40, 5½, 3, 12.

The following denominations are also in use:

3	barleycorns	= 1 inch.
4	inches	= 1 hand.
9	"	= 1 span.
21.888	"	= 1 sacred cubic.
6	feet	= 1 fathom.
3	"	= 1 pace.
5	paces	= 1 rod.
1.15⅓	statute miles	= 1 geographic mile.
3	geographic "	= 1 league = 3.458 statute miles.
60	" "	= 1 degree of latitude on a meridian or of longitude on the equator.
69.16	statute "	= 1 degree of latitude on a meridian or of longitude on the equator.
360	degrees	= the circumference of the earth.

NOTES. — 1. For the purpose of measuring cloth and other goods sold by the yard, the yard is divided into halves, fourths, eighths, and sixteenths. The old table of cloth measure is practically obsolete.

2. The inch of 3 barleycorns is used by shoemakers in measuring the length of the foot.

3. The hand of four inches is used in measuring the height of horses directly over the fore feet.

4. A span is the distance that can be reached, spanned, or measured between the end of the middle finger and the end of the thumb. Among sailors 8 spans are equal to 1 fathom.

5. The geographic mile is $\frac{1}{60}$ of $\frac{1}{360}$, or $\frac{1}{21600}$ of the distance round the center of the earth. It is a small fraction more than 1.15 statute miles. It is also called a nautical mile.

6. The length of a degree of latitude varies, being 68.72 miles at the equator, 68.9 to 69.05 miles in the middle latitudes, and 69.30 to 69.34 miles in the polar regions. The mean or average length, as stated in the table, is the standard recently adopted by the U. S. Coast Survey. A degree of longitude is greatest at the equator, where it is 69.16 miles, and it gradually decreases toward the poles, where it is 0.

METRIC LONG MEASURE.

338. The **Meter** is the metric *unit* of **Length.** It was intended to be $\frac{1}{10000000}$ part of the distance from the equator to either pole, but it does not exactly correspond with that length.

NOTE. — In this and the following tables of the metric system the principal units are in capitals, while others in common use are in black type.

TABLE.

	=	**1 millimeter**	= .03937079	in.
10 millimeters (mm)	=	**1 centimeter**	= .3937079	"
10 centimeters (cm)	=	1 decimeter	= 3.937079	"
10 decimeters (dm)	=	1 METER	= 39.37079	"
			= 1.0936	yd.
10 meters (m)	=	1 decameter	= 32.808992	ft.
10 decameters (Dm)	=	1 hectometer	= 19.927817	rd.
10 hectometers (Hm)	=	**1 kilometer**	= .6213824	mi.
10 kilometers (Km)	=	1 myriameter	= 6.213824	"

NOTE. — The measures chiefly used are the meter and kilometer. The meter, like the yard, is used in measuring cloth and short distances; the kilometer is used in measuring long distances.

UNIT EQUIVALENTS.

Mm.	Km.	Hm.	Dm.	m.	dm.	cm.	mm.
						1 =	10
					1 =	10 =	100
				1 =	10 =	100 =	1000
			1 =	10 =	100 =	1000 =	10000
		1 =	10 =	100 =	1000 =	10000 =	100000
	1 =	10 =	100 =	1000 =	10000 =	100000 =	1000000
1 =	10 =	100 =	1000 =	10000 =	100000 =	1000000 =	10000000

SCALE — uniformly 10.

SURVEYORS' LONG MEASURE.

339. The *unit* in **Land Surveying** is the **Gunter's Chain,** 4 rods or 66 feet long, consisting of 100 links, and the table is made up of divisors and multiples of this unit.

TABLE.

7.92	inches (in.)	= 1 link (l.).
25	links	= 1 rod (rd.).
4	rods, or 66 feet	= 1 chain (ch.).
80	chains	= 1 mile (mi.).

UNIT EQUIVALENTS.

			l.	in.
		rd.	1 =	7.92
	ch.	1 =	25 =	198
mi.	1 =	4 =	100 =	792
1 =	80 =	320 =	8000 =	63360

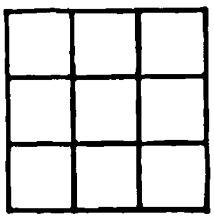
SCALE — ascending, 7.92, 25, 4, 80; descending, 80, 4, 25, 7.92.

NOTE. — Distances are usually taken in chains and links. In measuring city lots a steel tape 50 ft. long is generally used, and the measure is expressed in feet and in tenths of a foot. An engineer's chain used by civil engineers is 100 ft. long and consists of 100 links.

SQUARE MEASURE.

340. A **Square** is a figure having four equal sides and four equal corners or right angles.

341. **Area** or **Superficies** is the space or surface included within any given lines; as, the area of a square, of a board, etc.



1 square yard is a figure having four sides of 1 yard or 3 feet each, as shown in the diagram. Its contents are 3 × 3 = 9 square feet. Hence,

The contents or area of a square, or of any other figure having a uniform length and a uniform breadth, is found by multiplying the length by the breadth.

Thus, a square foot is 12 in. long and 12 in. wide, and the contents is 12 × 12 = 144 sq. in. A floor 20 ft. long and 10 ft. wide is a rectangle containing 20 × 10 = 200 sq. ft.

NOTE. — The measurements for computing area or surface are always taken in the denominations of linear measure.

342. **Square Measure** is used in computing areas or surfaces; as of land, boards, painting, etc. The *unit* is the area of a square whose side is the unit of length. Thus, the unit of square feet is 1 foot square; of square yards, 1 yard square, etc.

TABLE.

144	square inches (sq. in.)	=	1 square foot (sq. ft.).
9	" feet	=	1 " yard (sq. yd.).
$30\frac{1}{4}$	" yards	=	1 " rod (sq. rd.).
160	" rods	=	1 acre (A.).
640	acres	=	1 square mile (sq. mi.).

UNIT EQUIVALENTS.

sq. mi.	A.	sq. rd.	sq. yd.	sq. ft.	sq. in.
				1 =	144
			1 =	9 =	1296
		1 =	$30\frac{1}{4}$ =	$272\frac{1}{4}$ =	39204
	1 =	160 =	4840 =	43560 =	6272640
1 =	640 =	102400 =	3097600 =	27878400 =	4014489600

SCALE — ascending, 144, 9, $30\frac{1}{4}$, 160, 640; descending, 640, 160, $30\frac{1}{4}$, 9, 144.

Artificers estimate their work as follows: By the square foot — glazing and stonecutting. By the square yard — painting, plastering, paving, ceiling, and paper hanging. By the square of 100 square feet — flooring, partitioning, roofing, slating, and tiling. Bricklaying is estimated by the thousand bricks, by the square yard, and by the square of 100 square feet; also sometimes in cubic feet.

NOTES. — 1. In estimating the painting of moldings, cornices, etc., the measuring line is carried into all the moldings and cornices.

2. In estimating bricklaying by either the square yard or the square of 100 square feet, the work is understood to be 12 inches or $1\frac{1}{2}$ bricks thick.

3. A thousand shingles are estimated to cover 1 square, being laid 5 inches to the weather.

4. The terms *perch* or *pole* are sometimes used for square rod.

METRIC SQUARE MEASURE.

343. The **Square Meter** is the metric *unit* for measuring ordinary **Surfaces**; as floorings, ceilings, etc.

TABLE.

100 sq. millimeters (sq mm)	=	1 sq. centimeter	=	.155 + sq. in.
100 sq. centimeters (sq cm)	=	1 sq. decimeter	=	15.5 + sq. in.
100 sq. decimeters (sq dm)	=	1 SQ. METER (sq m)	=	1.196 + sq. yd.

METRIC LAND MEASURE.

344. The **Are** is the metric *unit* of **Land Measure**, and is a square whose side is 10 meters, equal to a *square decameter*, or 119.6 square yards.

TABLE.

1 centare (ca)	= 1 sq. meter	= 1.196034 sq. yd.
100 centares	= 1 ARE	= 119.6034 sq. yd.
100 ares (a)	= 1 hectare	= 2.47114 acres.
100 hectares (Ha)	= 1 sq. kilometer	= .3861 sq. mi.

NOTE. — The square kilometer is used in measuring the areas of countries, seas, etc.

UNIT EQUIVALENTS.

sq. Km.	Ha.	a.	sq. m. or ca.	sq. dm.	sq. cm.	sq. mm.
					1 =	100
				1 =	100 =	10000
			1 =	100 =	10000 =	1000000
		1 =	100 =	10000 =	1000000 =	100000000
	1 =	100 =	10000 =	1000000 =	100000000 =	10000000000
1 =	100 =	10000 =	1000000 =	100000000 =	10000000000 =	1000000000000

SCALE — uniformly 100.

SURVEYORS' SQUARE MEASURE.

345. Surveyors' Square Measure is used by surveyors in computing the area or contents of land.

TABLE.

625 square links (sq. l.)	= 1 pole (P.).
16 poles	= 1 square chain (sq. ch.).
10 square chains	= 1 acre (A.).
640 acres	= 1 square mile (sq. mi.).
36 square miles (6 miles square)	= 1 township (Tp.).

UNIT EQUIVALENTS.

Tp.	sq. mi.	A.	sq. ch.	P.	sq. l.
				1 =	625
			1 =	16 =	10000
		1 =	10 =	160 =	100000
	1 =	640 =	6400 =	102400 =	64000000
1 =	36 =	23040 =	230400 =	3686400 =	2304000000

SCALE — ascending, 625, 16, 10, 640, 36; descending, 36, 640, 10, 16, 625.

NOTE. — Canal and railroad engineers commonly use an engineers' chain, which consists of 100 links, each 1 foot long.

UNITED STATES LAND MEASURE.

346. The *unit* of **Land Measure** in the United States is the **Acre.** Measurements of land are commonly recorded in square miles, acres, and hundredths of an acre.

Government Lands are usually surveyed into rectangular tracts, bounded by lines conforming to the cardinal points of the compass.

A **Base Line** on a parallel of latitude, and a **Principal Meridian** intersecting it, are first established. Other lines are then run *six miles* apart, each way, as nearly as possible. The tracts thus formed are called **Townships**, and contain nearly 23040 acres. A line of townships extending north and south is called a **Range.** The ranges are designated by their number east or west of the principal meridian. The townships in each range are designated by their number north or south of the base line.

Since the earth's surface is convex, the principal meridians converge as they proceed northward. This tends to throw the townships and sections out of square, and necessitates occasional lines of offset, called "*correction lines.*"

Townships are subdivided into **Sections** and sections into **Half-Sections, Quarter-Sections, Half-Quarter-Sections, Quarter-Quarter-Sections,** and **Lots.**

DIAGRAM No. 1.

A TOWNSHIP.

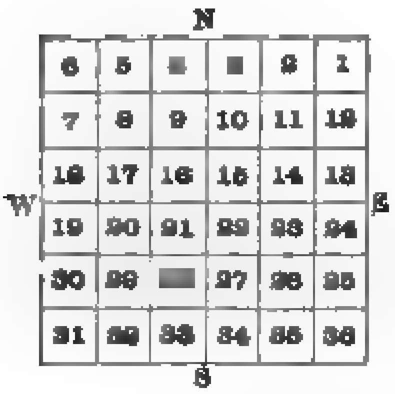


DIAGRAM No. 2.

A SECTION.

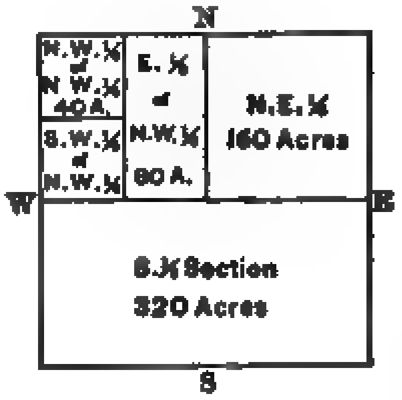


Diagram No. 1 shows the subdivisions of a *Township* into Sections, and how they are *numbered*, commencing at the N.E. corner.

Diagram No. 2 shows the subdivisions of a *Section*, on an enlarged scale, and how they are *named.*

TABLE.

6 mi. × 6 mi.	= 36 sq. mi.	= 23040	acres	= 1 Township.
1 " × 1 "	= 1 "	= 640	"	= 1 Section.
1 " × $\frac{1}{2}$ "	= $\frac{1}{2}$ "	= 320	"	= 1 Half-Section.
$\frac{1}{2}$ " × $\frac{1}{2}$ "	= $\frac{1}{4}$ "	= 160	"	= 1 Quarter-Section.
$\frac{1}{2}$ " × $\frac{1}{4}$ "	= $\frac{1}{8}$ "	= 80	"	= 1 Half-Quarter-Section.
$\frac{1}{4}$ " × $\frac{1}{4}$ "	= $\frac{1}{16}$ "	= 40	"	= 1 Quarter-Quarter-Section.

NOTE. — A **Lot** is a subdivision of a section, usually of irregular form, on account of bordering upon a navigable river or lake — containing as nearly as possible the area of a Quarter-Quarter-Section, and described as lot No. 1, 2, 3, etc., of a particular section. City and village plots are usually subdivided into *Blocks*, and these into smaller *Lots*.

FRENCH LAND MEASURE.

347. The old **French Linear**, or **Land Measure**, is still used to some extent in Louisiana, and in other French settlements in the United States.

TABLE.

12 lines = 1 inch.	6 feet = 1 toise.
12 inches = 1 foot.	32 toises = 1 arpent.

900 square toises = 1 square arpent.

NOTES. — 1. The *French foot* equals nearly 12.8 American inches.
2. The *arpent*, the old French name for *acre*, contains nearly $\frac{5}{6}$ of our acre.

SPANISH LAND MEASURE.

348. **Spanish Land Measures** are still used in Texas, New Mexico, and other Spanish settlements of the U. S.

The *unit* of length is the **Vara**, equal in Texas to $33\frac{1}{3}$ inches, in California to 33 inches, and in Mexico to 32.9927 inches.

Land is measured in square varas, labors, and square leagues.

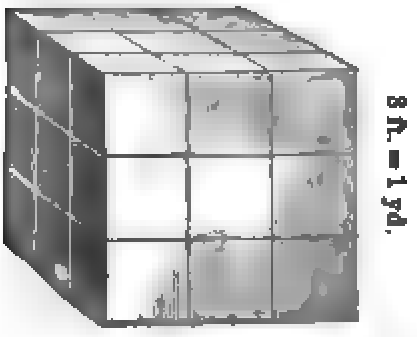
TABLE.

1000000 square varas	= 1 labor	= 177.136 acres (American).
25 labors	= 1 league	= 4428.4 acres "
1 acre		= 5645.376 square varas.

NOTE. — The *Spanish foot* = 11.11 + in. (Am.); 1 vara = $33\frac{1}{3}$ in. (Am.); 108 varas = 100 yards, and 1900.8 varas = 1 mile.

CUBIC MEASURE.

349. A **Cube** is a solid, or body, having six equal square sides or faces.



3 ft. = 1 yd.

350. **Solidity** is the matter or space contained within the bounding surfaces of a solid.

The measurements for solidity are always taken in the denominations of linear measure. If each side of a cube is 1 yard, or 3 feet, 1 foot in thickness of this cube will contain $3 \times 3 \times 1 = 9$ cubic feet; and the whole cube will contain $3 \times 3 \times 3 = 27$ cubic feet.

A solid, or body, may have the three dimensions all alike or all different. A body 4 feet long, 3 feet wide, and 2 feet thick contains $4 \times 3 \times 2 = 24$ cubic or solid feet.

The cubic or solid contents of a body is the product of the length, breadth, and thickness.

351. Cubic Measure, or **Solid Measure**, is used in computing the contents of solids, or bodies; as wood, stone, etc. The *unit* is the solidity of a cube whose side is the unit of length. Thus, the unit of cubic feet is a cube which measures 1 foot on each side; the unit of cubic yards is 1 cubic yard.

TABLE.

1728	cubic inches (cu. in.)	= 1 cubic foot (cu. ft.).
27	cubic feet	= 1 cubic yard (cu. yd.).
40	cubic feet of round timber, or	= 1 ton or load (T.).
50	" " hewn "	
16	cubic feet	= 1 cord foot (cd. ft.).
8	cord feet, or	= 1 cord of wood (Cd.).
128	cubic feet	
$24\frac{3}{4}$	cubic feet	= 1 perch of stone or masonry (Pch.).

SCALE—ascending, 1728, 27. The other numbers are not in a regular scale, but are merely so many times 1 foot. The unit equivalents, being fractional, are omitted.

NOTES.—1. A cubic yard of earth is called a load.

2. Railroad and transportation companies estimate light freight by the space it occupies in cubic feet, and heavy freight by weight.

3. A pile of wood 8 feet long, 4 feet wide, and 4 feet high, contains 1 cord; and a cord foot is 1 foot in length of such a pile.

4. A perch of stone or of masonry is $16\frac{1}{2}$ feet long, $1\frac{1}{2}$ feet wide, and 1 foot high.

5. Joiners, bricklayers, and masons make an allowance for windows, doors, etc., of one half the openings or vacant spaces. Bricklayers and masons, in estimating their work by cubic measure, make no allowance for the corners of the walls of houses, cellars, etc., but estimate their work by the *girt*, that is, the entire length of the wall on the *outside*.

6. Engineers, in making estimates for excavations and embankments, take the dimensions with a line or measure divided into feet and decimals of a foot. The computations are made in feet and decimals, and the results are reduced to cubic yards. In civil engineering, the cubic yard is the unit to which estimates for excavations and embankments are finally reduced.

7. In scaling or measuring timber for shipping or freighting, $\frac{1}{5}$ of the solid contents of round timber is deducted for waste in hewing or sawing. Thus, a log that will make 40 feet of hewn or sawed timber, actually contains 50 cubic feet by measurement; but its market value is only equal to 40 cubic feet of hewn or sawed timber. Hence, the cubic contents of 40 feet of round and 50 feet of hewn timber, as estimated for market, are identical.

METRIC CUBIC MEASURE.

52. The **Cubic Meter** is the metric *unit* for measuring ordi-
y **Solids**; as excavations, embankments, etc.

TABLE.

1000 cu. millimeters (cu mm)	= 1 cu. centimeter	=	.061 + cu. in.
1000 cu. centimeters (cu cm)	= 1 cu. decimeter	=	61.026 + " "
1000 cu. decimeters (cu dm)	= 1 CU. METER	=	35.316 + cu. ft. or 1.308 cu. yd.

UNIT EQUIVALENTS.

cu. m.	cu. dm.	cu. cm.	cu. mm.
		1 =	1000
	1 =	1000 =	1000000
1 =	1000 =	1000000 =	1000000000

CALE — uniformly 1000.

METRIC WOOD MEASURES.

53. The **Stere** is the *unit* of **Wood** or **Solid Measure**, and is
al to a *cubic meter*, or .2759 cord.

TABLE.

	1 decistere	= 3.531 + cu. ft.
10 decisteres (dst)	= 1 STERE	= 35.316 + cu. ft.
10 steres (st)	= 1 decastere (Dst)	= 13.079 + cu. yd.

UNIT EQUIVALENTS.

Dst.	st.	dst.
	1 =	10
1 =	10 =	100

CALE — uniformly 10.

MEASURES OF CAPACITY.

54. **Capacity** signifies extent of room or space.

TE. — Measures of capacity are all cubic measures, solidity and capacity being referred
ferent units, as will be seen by comparing the tables. Measures of capacity may be
rly subdivided into two classes, — Measures of Liquids and Measures of Dry
ances.

LIQUID MEASURE.

55. **Liquid Measure,** also called Wine Measure, is used in
suring liquids. The *unit* is the **Gallon**, and the table is
e up of its divisors and multiples.

TABLE.

4 gills (gi.)	= 1 pint (pt.).
2 pints	= 1 quart (qt.).
4 quarts	= 1 gallon (gal.).
31½ gallons	= 1 barrel (bbl.).
2 barrels, or 63 gal.	= 1 hogshead (hhd.).

UNIT EQUIVALENTS.

hhd.	bbl.	gal.	qt.	pt.	gi.
				1 =	4
			1 =	2 =	8
		1 =	4 =	8 =	32
	1 =	31½ =	126 =	252 =	1008
1 =	2 =	63 =	252 =	504 =	2016

SCALE — ascending, 4, 2, 4, 31½, 2; descending, 2, 31½, 4, 2, 4.

The following denominations are also in use:

36 gallons	= 1 barrel of beer.
54 " or 1½ barrels	= 1 hogshead of beer.
42 "	= 1 tierce.
2 hogsheads, or 126 gallons	= 1 pipe or butt.
2 pipes, or 4 hogsheads	= 1 tun.

NOTES. — 1. The denominations, barrel and hogshead, are used in estimating the capacity of cisterns, reservoirs, vats, etc.

2. The tierce, hogshead, pipe, butt, and tun are the names of casks, and do not express any fixed or definite measures. They are usually gauged and have their capacities in gallons marked on them. Several of these denominations are still in use in England.

3. Ale or beer measure, formerly used in measuring beer, ale, and milk is almost entirely discarded.

DRY MEASURE.

356. **Dry Measure** is used in measuring articles not liquid; as grain, fruit, salt, roots, ashes, etc. The *unit* is the **Bushel**, of which all the other denominations in the table are divisors.

TABLE.

2 pints (pt.)	= 1 quart (qt.).
8 quarts	= 1 peck (pk.).
4 pecks	= 1 bushel (bu.).

UNIT EQUIVALENTS.

bu.	pk.	qt.	pt.
		1 =	2
	1 =	8 =	16
1 =	4 =	32 =	64

SCALE — ascending, 2, 8, 4; descending, 4, 8, 2.

NOTES. — 1. In England, 8 bu. of 70 lb. each (used in measuring grain) are called a *irter*. The weight of the English quarter is $\frac{1}{4}$ of a long ton.

2. Grain and some other commodities are sold by even full or *stricken measure*, and such cases the "measure is to be stricken with a round stick or roller, straight, and of same diameter from end to end."

3. Coal, ashes, marl, manure, corn in the ear, fruit and roots are sold by *heap measure*. e bushel, heap measure, is the Winchester bushel heaped in the form of a cone, which ie must be $19\frac{1}{2}$ inches in diameter (= to the outside diameter of the standard bushel asure), and at least 6 inches high. A bushel, heap measure, contains 2747.7167 cubic hes, or 597.2967 cubic inches more than a bushel stricken measure. Since 1 peck cons $\frac{2150.42}{4} = 537.605$ cubic inches, the bushel, heap measure, contains 59.6917 cubic inches re than 5 pecks. As this is about 1 bu. 1 pk. $1\frac{3}{4}$ pt., it is sufficiently accurate in practice, call 5 pecks stricken measure a heaped bushel.

4. A standard bushel, stricken measure, is commonly estimated at 2150.42 cubic inches. e old English standard bushel from which the United States standard bushel was derived, s kept at Winchester, England; hence the name.

5. The wine and dry measures of the same denomination are of different capacities. e exact and the relative size of each may be seen by the following table.

357. COMPARATIVE TABLE OF MEASURES OF CAPACITY.

	Cubic in. in one gallon.	Cubic in. in one quart.	Cubic in. in one pint.	Cubic in. in one gill.
quid measure	231	$57\frac{3}{4}$	$28\frac{7}{8}$	$7\frac{7}{32}$
ry measure ($\frac{1}{2}$ pk.) . .	$268\frac{4}{5}$	$67\frac{1}{5}$	$33\frac{3}{5}$	$8\frac{2}{5}$

NOTE. — The beer gallon of 282 inches is retained in use only by custom.

EQUIVALENTS.

231 cu. in. = 1 gal.
2150.42 cu. in. = 1 bu.
4 bu. = 5 cu. ft. (nearly).
4 heaped bu. = 5 stricken bu.
$7\frac{1}{2}$ gal. = 1 cu. ft. (nearly).
7 cu. ft. corn in ear = 3 bu. shelled corn.

METRIC CAPACITY.

358. The **Liter** is the metric *unit* of **Capacity**, both of Liquid id of Dry Measures, and is a vessel whose volume is equal to cube whose edge is *one tenth* of a *meter*, equal to 1.05673 qt. iquid Measure, and .9081 qt. Dry Measure.

TABLE.

10 milliliters (^{ml}) = 1 centiliter.
10 centiliters (^{cl}) = 1 deciliter.
10 deciliters (^{dl}) = 1 LITER
10 liters (^{l}) = 1 decaliter.
10 decaliters (^{Dl}) = 1 **hectoliter**.
10 hectoliters (^{Hl}) = 1 kiloliter, or stere.
10 kiloliters (^{Kl}) = 1 myrialiter (^{Ml}).

UNIT EQUIVALENTS.

						cl.	ml.
					dl.	1 =	10
				l.	1 =	10 =	100
			Dl.	1 =	10 =	100 =	1000
		Hl.	1 =	10 =	100 =	1000 =	10000
	Kl.	1 =	10 =	100 =	1000 =	10000 =	100000
Ml.	1 =	10 =	100 =	1000 =	10000 =	100000 =	1000000
1 =	10 =	100 =	1000 =	10000 =	100000 =	1000000 =	10000000

SCALE — uniformly 10.

NOTE.—The measures commonly used are the liter and hectoliter. The liter is very nearly a quart; it is used in measuring milk, wine, etc., in moderate quantities. The hectoliter is about 2 bu. 3¼ pk.; it is used in measuring grain, fruit, roots, etc., in large quantities.

359. EQUIVALENTS IN UNITED STATES MEASURES.

Metric Denominations.		Cubic Measure.		Dry Measure.		Wine Measure.
1 milliliter	=	1 cu. centimeter	=	.061 cu. in.	=	.27 fluid dr.
1 centiliter	=	10 cu. centimeters	=	.6102 cu. in.	=	.338 fluid oz.
1 deciliter	=	$\frac{1}{10}$ cu. decimeter	=	6.1022 cu. in.	=	.845 gill.
1 LITER	=	1 cu. decimeter	=	.908 quart	=	1.0567 qt.
1 decaliter	=	10 cu. decimeters	=	9.08 quarts	=	2.6417 gal.
1 hectoliter	=	$\frac{1}{10}$ cubic meter	=	2.8372+ bu.	=	26.417 gal.
1 kiloliter	=	1 cubic meter	=	28.372+ bu.	=	264.17 gal.
1 myrialiter	=	10 cubic meters	=	283.72+ bu.	=	2641.7+ gal.

MEASURES OF WEIGHT.

360. **Weight** is the measure of the quantity of matter a body contains, determined by the force of gravity.

NOTE.—The process by which the quantity of matter or the force of gravity is obtained is called *weighing;* and consists in comparing the thing to be weighed with some conventional standard.

Three scales of weight are used in the United States; namely, Troy, avoirdupois, and apothecaries'.

NOTES.—1. The term *avoirdupois* is derived from the French, *avoir du poids, to have weight.* The term *Troy* is derived from *Troyes*, the name of a town in France, where the weight was first used in Europe.

2. *Pound* is from the Latin *pendo, bend*, or *weigh*. *Ounce* is from the Latin *uncia, a twelfth*, the ounce being a twelfth of a pound. *Pennyweight* derives its name from the fact that it was the weight of an old English penny. The *grain* of wheat was formerly the standard of all weights in England.

TROY WEIGHT.

361. Troy Weight is used in weighing gold, silver, and jewels; in philosophical experiments, and generally where great accuracy is required. The *unit* is the **Pound**, and of this all the other denominations in the table are divisors.

TABLE.

24 grains (gr.)	= 1 pennyweight (pwt. or dwt.).
20 pennyweights	= 1 ounce (oz.).
12 ounces	= 1 pound (lb.).

UNIT EQUIVALENTS.

lb.	oz.	pwt.	gr.
		1 =	24
	1 =	20 =	480
1 =	12 =	240 =	5760

SCALE — ascending, 24, 20, 12; descending, 12, 20, 24.

NOTES. — 1. Troy weight is sometimes called *goldsmiths' weight*.

2. In weighing diamonds, pearls, and other jewels, the unit commonly employed is the carat, equal to 4 carat grains or 3.168 Troy grains. The term *carat* is also used to express the number of parts in 24 that are pure gold. Thus gold that is 12 carats fine is $\frac{12}{24}$ pure gold and $\frac{12}{24}$ alloy.

AVOIRDUPOIS WEIGHT.

362. Avoirdupois Weight is used for all the ordinary purposes of weighing. The *unit* is the **Pound**, and the table is made up of its divisors and multiples.

TABLE.

16 drams (dr.)	= 1 ounce (oz.).
16 ounces	= 1 pound (lb.).
100 pounds	= 1 hundredweight (cwt.).
20 cwt., or 2000 lb.	= 1 ton (T.).

UNIT EQUIVALENTS.

T.	cwt.	lb.	oz.	dr.
			1 =	16
		1 =	16 =	256
	1 =	100 =	1600 =	25600
1 =	20 =	2000 =	32000 =	512000

SCALE — ascending, 16, 16, 100, 20; descending, 20, 100, 16, 16.

NOTE. — The *long* or *gross ton*, hundredweight, and quarter were formerly in common use; but they are now seldom used, except in estimating English goods at the U. S. custom houses, in freighting and wholesaling coal from the Pennsylvania mines, and in the wholesale iron and plaster trade.

LONG TON TABLE.

16 ounces (oz.) = 1 pound.
28 pounds = 1 quarter (qr.).
4 qr. = 112 lb. = 1 hundredweight (cwt.).
20 cwt. = 2240 lb. = 1 ton (T.).

SCALE — ascending, 28, 4, 20; descending, 20, 4, 28.

The following denominations are also in use:

14 pounds			= 1 stone.
100	"	butter	= 1 firkin.
100	"	grain or flour	= 1 cental.
100	"	dry fish	= 1 quintal.
100	"	nails	= 1 keg.
196	"	flour	= 1 barrel.
200	"	pork or beef	= 1 barrel.
280	"	salt at N. Y. S. Works	= 1 barrel.
56	"	salt at N. Y. S. Works	= 1 bushel.
240	"	lime	= 1 cask.

363. The number of avoirdupois pounds in a bushel, as fixed by statute, varies in different states and with different articles. The following are the weights established in most of the states:

TABLE OF AVOIRDUPOIS POUNDS IN BUSHELS.

COMMODITIES.	Weight in most states.	Weight in other states.	COMMODITIES.	Weight in most states.	Weight in other states.
Apples	50	44–57	Hair	8	11
Apples, dried . . .	24	22–28	Hemp seed	44	No excep.
Barley	48	46–50	Hungarian seed . .	50	48
Beans, castor . . .	46	45–62	Malt, barley	38	30–35
Beans, white . . .	60	62	Millet seed	50	48
Bluegrass seed. . .	14	No excep.	Oats	32	26–36
Bran	20	No excep.	Onions	57	48–56
Buckwheat	52	40–56	Pease	60	No excep.
Clover seed	60	62–64	Potatoes, Irish . .	60	56
Corn, ear	70	54–72	Potatoes, sweet . .	55	46–60
Corn, shelled . . .	56	52	Rye	56	54–60
Corn meal	50	46–48	Timothy seed . . .	45	42–60
Cotton seed	32	28–40	Turnips	55	42–60
Flaxseed	56	55	Wheat	60	No excep.

APOTHECARIES' WEIGHT.

364. Apothecaries' Weight is used by apothecaries and physicians in weighing medicines for prescriptions; but medicines are bought and sold by avoirdupois weight. The *unit* is the **Pound,** of which the other denominations are divisors.

TABLE.

20 grains (gr.)	= 1 scruple (sc. or ℈).
3 scruples	= 1 dram (dr. or ʒ).
8 drams	= 1 ounce (oz. or ℥).
12 ounces	= 1 pound (lb. or ℔).

UNIT EQUIVALENTS.

℔.		oz.		dr.		sc.		gr.
						1	=	20
				1	=	3	=	60
		1	=	8	=	24	=	480
1	=	12	=	96	=	288	=	5760

SCALE — ascending, 20, 3, 8, 12; descending, 12, 8, 3, 20.

APOTHECARIES' FLUID MEASURE.

365. The measures for fluids, used in compounding medicines, and putting them up for market, are as follows:

TABLE.

60 minims (♏)	= 1 fluidrachm (fʒ).
8 fluidrachms	= 1 fluidounce (f℥).
16 fluidounces	= 1 pint (O).
8 pints	= 1 gallon (Cong.).

UNIT EQUIVALENTS.

Cong.		O.		f℥		fʒ		♏
						1	=	60
				1	=	8	=	480
		1	=	16	=	128	=	7680
1	=	8	=	128	=	2048	=	61440

SCALE — ascending, 60, 8, 16, 8; descending, 8, 16, 8, 60.

COMPARATIVE TABLE OF WEIGHTS.

	Troy.	Avoirdupois.	Apoth.
1 pound =	5760 gr. =	7000 gr. =	5760 gr.
1 ounce =	480 gr. =	437.5 gr. =	480 gr.
	175 lb. =	144 lb. =	175 lb.

METRIC WEIGHT.

366. The **Gram** is the metric *unit* of **Weight**, and is equal to the weight of a cube of distilled water, the edge of which is *one hundredth* of a *meter*, equal to 15.432 Troy grains.

TABLE.

10 milligrams (mg)	= 1 centigram	=	.15432 + gr. Troy.
10 centigrams (cg)	= 1 decigram	=	1.54324 + " "
10 decigrams (dg)	= 1 GRAM	=	15.43248 + " "
10 grams (g)	= 1 decagram	=	.35273 + oz. Avoir.
10 decagrams (Dg)	= 1 hectogram	=	3.52739 + " "
10 hectograms (Hg)	= 1 kilogram,	=	2.20462 + lb. "
10 kilograms (Kg)	= 1 myriagram	=	22.04621 + " "
10 myriagrams (Mg)	= 1 quintal	=	220.46212 + " "
10 quintals	= 1 {tonneau, or ton}	=	2204.62125 " "

UNIT EQUIVALENTS.

T.	Q.	Mg.	Kg.	Hg.	Dg.	g.	dg.	cg.	mg.
								1 =	10
							1 =	10 =	100
						1 =	10 =	100 =	1000
					1 =	10 =	100 =	1000 =	10000
				1 =	10 =	100 =	1000 =	10000 =	100000
			1 =	10 =	100 =	1000 =	10000 =	100000 =	1000000
		1 =	10 =	100 =	1000 =	10000 =	100000 =	1000000 =	10000000
	1 =	10 =	100 =	1000 =	10000 =	100000 =	1000000 =	10000000 =	100000000
1 =	10 =	100 =	1000 =	10000 =	100000 =	1000000 =	10000000 =	100000000 =	1000000000

SCALE — uniformly 10.

NOTE.—The *gram* is used in mixing medicines, in weighing the precious metals, and whenever great exactness is required. The *kilogram*, or *kilo* is the usual weight for groceries and coarse articles generally; it is very nearly $2\frac{1}{5}$ pounds avoirdupois. The *tonneau* is used for weighing hay and other heavy articles; it is about 204 lb. more than our ton.

EQUIVALENTS.

367. Units of the **Common System** may be readily changed to units of the **Metric System** by the aid of the following table:

TABLE.

1 inch	= 2.54 centimeters.	1 cu. inch	= 16.39 cu. centimeters.
1 foot	= 30.48 centimeters.	1 cu. foot	= 28320 cu. centimeters.
1 yard	= .9144 meter.	1 cu. yard	= .7646 cu. meter.
1 rod	= 5.029 meters.	1 cord	= 3.625 steres.
1 mile	= 1.6093 kilometers.	1 fl. ounce	= 2.958 centiliters.
1 sq. inch	= 6.4528 sq. centimeters.	1 gallon	= 3.786 liters.
1 sq. foot	= 929 sq. centimeters.	1 bushel	= .35242 hectoliter.
1 sq. yd.	= .8361 sq. meter.	1 Troy gr.	= 64.8 milligrams.
1 sq. rod	= 25.29 centares.	1 Troy lb.	= .373 kilo.
1 acre	= 40.47 ares.	1 av. lb.	= .4536 kilo.
1 sq. mile	= 259 hectares = 2.59 sq.Km.	1 ton	= .907 tonneau.

Money and Currencies.

368. **Money** is the commodity adopted to serve as the universal equivalent or measure of value of all other commodities, and for which individuals readily exchange their surplus products or their services.

Coin is metal struck, stamped, or pressed with a die, to give it a legal, fixed value, for the purpose of circulating as money. The coins of civilized nations consist of gold, silver, copper, nickel, and bronze.

A **Mint** is a place in which the coin of a country or government is manufactured. In all civilized countries mints and coinage are under the exclusive direction and control of government.

An **Alloy** is a metal compounded with another metal of greater value. In coinage, the less valuable or *baser metal* is not reckoned of any value.

Note. — Gold and silver, in their pure state, are too soft and flexible for coinage; hence they are hardened by compounding them with an alloy of baser metal, while their color and other valuable qualities are not materially impaired.

An **Assayer** is a person who determines the composition and consequent value of alloyed gold and silver.

The **fineness** of gold is estimated by *carats*, as follows:

Any mass or quantity of gold, either pure or alloyed, is divided into 24 equal parts, and each part is called a *carat*.

Fine gold is pure, and is 24 carats fine.

Alloyed gold is as many carats fine as it contains parts in 24 of fine or pure gold. Thus, gold 20 carats fine contains 20 parts or carats of fine gold, and 4 parts or carats of alloy.

An **Ingot** is a small mass or bar of gold or silver, intended either for coinage or exportation. Ingots for exportation usually have the assayer's or mint value stamped upon them.

Bullion is uncoined gold or silver.

Bank Bills or **Bank Notes** are bills or notes issued by a banking company, and are payable to the bearer in gold or silver, at the bank, on demand. They are substitutes for coin, but are not legal tender in payment of debts or other obligations.

Treasury Notes are notes issued by the general government, and are payable to the bearer, on demand.

Currency is coin, bank bills, treasury notes, and other substitutes for money, employed in trade and commerce.

A **Circulating Medium** is the currency or money of a country or government.

A **Decimal Currency** is a currency whose denominations increase and decrease according to the decimal scale.

UNITED STATES MONEY.

369. The currency of the **United States** is decimal, and is sometimes called *Federal Money*. The *unit* is the **Dollar**, and all the other denominations are divisors or multiples of it.

TABLE.

10 mills (m.)	=	1 cent (ct.).
10 cents	=	1 dime (d.).
10 dimes	=	1 dollar ($).
10 dollars	=	1 eagle (E.).

UNIT EQUIVALENTS.

E.	$	d.	ct.	m.
			1 =	10
		1 =	10 =	100
	1 =	10 =	100 =	1000
1 =	10 =	100 =	1000 =	10000

SCALE — uniformly 10.

NOTE. — Federal Money was adopted by Congress in 1786. The character $ is supposed to be a contraction of U. S. (United States), the U being placed upon the S.

The coins of the United States with their weight and fineness as established by the coinage acts in force June 30, 1893, are as follows:

Coins.	Weight.	Fineness.
Double Eagle (gold)	516 grains	.900
Eagle (gold)	258 "	.900
Half Eagle (gold)	129 "	.900
Quarter Eagle (gold)	64.5 "	.900
Silver Dollar (silver)	412.5 "	.900
Half Dollar (silver)	192.9 "	.900
Quarter Dollar (silver) . . .	96.45 "	.900
Dime (silver)	38.4 "	.900
Five-cent piece (nickel) . . .	77.16 "	.75 copper, .25 nickel.
One-cent piece (bronze) . .	48 "	.95 copper, .05 tin and zinc.

NOTES. — The double eagle = $ 20, the eagle $ 10, and the half eagle $ 5. Bank bills are issued in denominations of $ 1, $ 2, $ 5, $ 10, $ 20, $ 50, $ 100, $ 500, and $ 1000.

CANADA MONEY.

370. The currency of the **Dominion of Canada** is decimal, and the table and denominations are the same as those of the United States money.

NOTE. — The currency of the whole Dominion of Canada was made uniform July 1, 1871. Before the adoption of the decimal system, pounds, shillings, and pence were used.

COINS. — The *gold coins* used in Canada are the British sovereign, worth $ 4.8665, and the half-sovereign.

The *bronze coin* is the cent.

The *silver coins* are the 50-cent piece, 25-cent piece, 10-cent piece, and 5-cent piece. The 20-cent piece is no longer coined.

The intrinsic value of the 50-cent piece in United States money is about 46.2 cents; of the 25-cent piece, 23.1 cents; of the 10-cent piece $ .092; of the 5-cent piece $ .046 and of the 1-cent piece $ .01. In ordinary business transactions Canadian coins pass the same as United States coins of the same denomination.

Government Standard. The silver coins consist of 925 parts (.925) pure silver and 75 parts (.075) copper. That is, they are 925 fine.

ENGLISH MONEY.

371. English or Sterling Money is the currency of Great Britain. The *unit* is the **Pound Sterling,** and all the other denominations are divisors of this unit.

TABLE.

		U. S. Value.
4 farthings (far. or qr.)	= 1 penny (d.)	= $ 0.0202.
12 pence	= 1 shilling (s.)	= $ 0.2433.
20 shillings	= 1 pound or sovereign (£ or sov.)	= $ 4.8665.

UNIT EQUIVALENTS.

			d.	far.
		s.	1 =	4
	£ or sov.	1 =	12 =	48
	1 =	20 =	240 =	960

SCALE — ascending, 4, 12, 20; descending, 20, 12, 4.

NOTES. — 1. Farthings are generally expressed as fractions of a penny; thus, 1 far., sometimes called 1 quarter (qr.) = ¼*d.*; 3 far. = ¾*d.*

2. The old /, the original abbreviation for shillings, was formerly written between shillings and pence, and *d.*, the abbreviation for pence, was omitted. Thus, 2*s.* 6*d.* was written 2/6. A straight line is now used in place of the /, and shillings are written on the left of it and pence on the right. Thus, 2/6, 10/3, etc.

COINS. — The *gold coins* are the sovereign and the half-sovereign.

The *silver coins* are the crown (= 5*s.* = $ 1.216), the half-crown (= 2*s.* 6*d.* = $.608), the florin (= 2*s.* = $.486), the shilling, the sixpence, the fourpence, and the threepence.

The *copper coins* are the penny (= $ .02), the half-penny, and the farthing.

NOTE. — The guinea (= 21*s.* = $ 5.11) and the half-guinea (= 10*s.* 6*d.* sterling = $ 2.555) are old gold coins, and are no longer coined.

Government Standard. The standard fineness of English gold coin is 11 parts pure gold and 1 part alloy; that is, it is 22 carats fine. The

standard fineness of silver coin is 11 oz. 2 pwt. (= 11.1 oz.) pure silver to 18 pwt. (= .9 oz.) alloy. Hence the silver coins are 11 oz. 2 pwt. fine; that is, 11 oz. 2 pwt. pure silver in 1 lb. standard silver. This standard is 37 parts ($\frac{37}{40}$ = .925) pure silver and 3 parts ($\frac{3}{40}$ = .075) copper.

NOTE. — A pound of English standard gold is equal in value to 14.2878 lb. = 14 lb. 8 oz. 9 pwt. 1.727 gr. of silver.

FRENCH MONEY.

372. The currency of **France** is decimal currency. The *unit* is the **Franc,** of which the other denominations are divisors.

TABLE.

10 millimes = 1 centime = $ .00193.
100 centimes = 1 franc = $ .193.

SCALE — uniformly 10.

COINS. — The *gold coins* are the 40-, 20-, 10-, and 5-franc pieces.

The *silver coins* are the 5-, 2-, and 1-franc, the 50- and 20-centime pieces.

The *bronze coins* are the 10-, 5-, 2-, and 1-centime pieces.

GERMAN MONEY.

373. The currency of **Germany** is decimal, the *unit* being the **Reichsmark** or **Mark.**

TABLE.

100 pfennigs = 1 mark = $ .2385.

COINS. — The *gold coins* are the 20-, 10-, and 5-mark pieces.

The *silver coins* are the 2-mark and 1-mark, and the 20-pfennig pieces.

The *nickel coins* are the 10-pfennig and 5-pfennig pieces, and the copper coins are the 2-pfennig and 1-pfennig pieces.

374. COMPARATIVE TABLE OF MONEYS.

English.	U. S.	French.	U. S.
1 penny (d.)	= $0.0202 +	1 centime (ct.)	= $0.00193.
1 shilling (s.)	= .2433 +	1 decime (dc.)	= 0.0193.
1 florin (fl.)	= .4866 +	1 franc (fr.)	= .193.
1 sovereign (sov.)	= 4.8665	German.	
		1 mark (mk.)	= 0.2385.

375. The Act of 1873, provides that the value of foreign coin, as expressed in United States money, shall be that of the pure metal of such coin of standard value. The following table is taken from the estimate made by the Director of the Mint, July 1, 1895:

VALUES OF FOREIGN COINS.

Country.	Standard.	Monetary unit.
Argentine Republic	Gold and silver	Peso
Austria-Hungary	Gold	Crown
Belgium	Gold and silver	Franc
Bolivia	Silver	Boliviano
Brazil	Gold	Milreis
British Possessions N. A. (except Newfoundland).	Gold	Dollar
Central Amer. States—		
Costa Rica Guatemala Honduras Nicaragua Salvador	Silver	Peso
Chile	Gold and silver	Peso
China	Silver	Tael: Shanghai Haikwan (Customs). Tientsin Chefoo
Colombia	Silver	Peso
Cuba	Gold and silver	Peso
Denmark	Gold	Crown
Ecuador	Silver	Sucre
Egypt	Gold	Pound (100 piasters)
Finland	Gold	Mark
France	Gold and silver	Franc
German Empire	Gold	Mark
Great Britain	Gold	Pound sterling
Greece	Gold and silver	Drachma
Haiti	Gold and silver	Gourde
India	Silver	Rupee
Italy	Gold and silver	Lira
Japan	Gold and silver[1]	Yen: Gold Silver
Liberia	Gold	Dollar
Mexico	Silver	Dollar
Netherlands	Gold and silver	Florin
Newfoundland	Gold	Dollar
Norway	Gold	Crown
Persia	Silver	Kran
Peru	Silver	Sol
Portugal	Gold	Milreis
Russia	Silver[2]	Ruble: Gold Silver
Spain	Gold and silver	Peseta
Sweden	Gold	Crown
Switzerland	Gold and silver	Franc
Tripoli	Silver	Mahbub of 20 piasters
Turkey	Gold	Piaster
Venezuela	Gold and silver	Bolivar

[1] Gold the nominal standard. Silver practically the standard.

[2] Silver the nominal standard. Paper, the actual currency, the depreciation of which is measured by the gold standard.

VALUES OF FOREIGN COINS.

Value in terms of U. S. gold dollar.	Coins.
$ 0.965	Gold: argentine ($ 4.824) and ½ argentine. Silver: peso and divisions.
.203	Gold: former system — 4 florins ($ 1.929), 8 florins ($ 3.858), ducat ($2.287), and 4 ducats ($ 9.158). Silver: 1 and 2 florins. Gold: present system — 20 crowns ($ 4.052); 10 crowns ($ 2.026).
.193	Gold: 10 and 20 francs. Silver: 5 francs.
.486	Silver: boliviano and divisions.
.546	Gold: 5, 10, and 20 milreis. Silver: ½, 1, and 2 milreis.
1.00	
.486	Silver: peso and divisions.
.912	Gold: escudo ($ 1.824), doubloon ($ 4.561), and condor ($ 9.123). Silver: peso and divisions.
.718	
.800	
.761	
.751	
.486	Gold: condor ($ 9.647) and double-condor. Silver: peso.
.926	Gold: doubloon ($ 5.017). Silver: peso.
.268	Gold: 10 and 20 crowns.
.486	Gold: condor ($ 9.647) and double-condor. Silver: sucre and divisions.
4.943	Gold: pound (100 piasters), 5, 10, 20, and 50 piasters. Silver: 1, 2, 5, 10, and 20 piasters.
.193	Gold: 20 marks ($ 3.859), 10 marks ($ 1.93).
.193	Gold: 5, 10, 20, 50, and 100 francs. Silver: 5 francs.
.238	Gold: 5, 10, and 20 marks.
4.866½	Gold: sovereign (pound sterling) and ½ sovereign.
.193	Gold: 5, 10, 20, 50, and 100 drachmas. Silver: 5 drachmas.
.965	Silver: gourde.
.231	Gold: mohur ($ 7.105). Silver: rupee and divisions.
.193	Gold: 5, 10, 20, 50, and 100 lire. Silver: 5 lire.
.997	Gold: 1, 2, 5, 10, and 20 yen.
.524	Silver: yen.
1.00	
.523	Gold: dollar ($ 0.983), 2½, 5, 10, and 20 dollars. Silver: dollar (or peso) and divisions.
.402	Gold: 10 florins. Silver: ½, 1, and 2½ florins.
1.014	Gold: 2 dollars ($ 2.027).
.268	Gold: 10 and 20 crowns.
.089	Gold: ½, 1, and 2 tomans ($ 3.409). Silver: ¼, ½, 1, 2, and 5 krans.
.486	Silver: sol and divisions.
1.08	Gold: 1, 2, 5, and 10 milreis.
.772	Gold: imperial ($ 7.718), and ½ imperial[1] ($ 3.86).
.389	Silver: ¼, ½, and 1 ruble.
.193	Gold: 25 pesetas. Silver: 5 pesetas.
.268	Gold: 10 and 20 crowns.
.193	Gold: 5, 10, 20, 50, and 100 francs. Silver: 5 francs.
.438	
.044	Gold: 25, 50, 100, 250, and 500 piasters.
.193	Gold: 5, 10, 20, 50, and 100 bolivars. Silver: 5 bolivars.

[1] Coined since January 1, 1886. Old half-imperial = $ 3.986.

Measure of Time.

376. Time is the measure of duration. The *unit* is the **Day,** and the table is made up of its divisors and multiples.

TABLE.

60 seconds (sec.)	= 1 minute (min.).
60 minutes	= 1 hour (h.).
24 hours	= 1 day (da.).
7 days,	= 1 week (wk.).
365 days or 52 wk. 1 da.	= 1 common year (yr.).
366 days	= 1 leap year (yr.).
12 calendar months	= 1 year (yr.).
100 years	= 1 century (C.).

UNIT EQUIVALENTS.

yr.	mo.	wk.	da.	h.	min.	sec.
					1 =	60
				1 =	60 =	3600
			1 =	24 =	1440 =	86400
		1 =	7 =	168 =	10080 =	604800
1 =	12 =		365 =	8760 =	525600 =	31536000
			366 =	8784 =	527040 =	31622400

Scale — ascending, 60, 60, 24, 7; descending, 7, 24, 60, 60.

The calendar year is divided as follows:

No. of month.	Season.	Names of months.	Abbreviations.	No. of days.
1	Winter,	January,	Jan.	31
2		February,	Feb.	28 or 29
3		March,	Mar.	31
4	Spring,	April,	Apr.	30
5		May,	——	31
6		June,	Jun.	30
7	Summer,	July,	——	31
8		August,	Aug.	31
9		September,	Sept.	30
10	Autumn,	October,	Oct.	31
11		November,	Nov.	30
12	Winter	December,	Dec.	31

The number of days in each calendar month may be easily remembered by committing the following lines to memory:

> "Thirty days have September,
> April, June, and November;
> All the rest have thirty-one,
> Save February, which alone
> Has twenty-eight; and one day more
> We add to it one year in four."

NOTE.—1. In most business transactions 30 days are called 1 month. For many purposes 4 weeks constitute a month.

2. The *civil day* begins and ends at 12 o'clock, midnight. The *astronomical day*, used by astronomers in dating events, begins and ends at 12 o'clock, noon. The civil year is composed of civil days.

3. A.M. (ante-meridian) is used to denote the time between midnight and noon; M. (meridian) to denote noontime; and P.M. (post-meridian) to denote the time between noon and midnight.

BISSEXTILE OR LEAP YEAR.

377. The period of time required by the sun to pass from one vernal equinox to another, called the vernal or tropical year, is exactly 365 da. 5 h. 48 min. 49.7 sec. This is the true year, and it exceeds the common year by 5 h. 48 min. 49.7 sec. (not quite a quarter of a day).

If 365 days are reckoned as 1 year, the time lost in the calendar will be

In 1 yr., 5 h. 48 min. 49.7 sec.
" 4 " 23 " 15 " 18.8 "

The time thus lost in 4 years will lack only 44 min. 41.2 sec. of 1 entire day. Hence,

If every fourth year is reckoned as leap year, the time *gained* in the calendar will be,

In 4 yr., 44 min. 41.2 sec.
" 100 " $(= 25 \times 4$ yr.$)$ 18 h. 37 " 10 "

The time thus gained in 100 years will lack only 5 h. 22 min. 50 sec. of 1 day. Hence,

If every fourth year is reckoned as leap year, the centennial years excepted, the time *lost* in the calendar will be,

In 100 yr., 5 h. 22 min. 50 sec.
" 400 " 21 " 31 " 20 "

The time thus lost in 400 years lacks only 2 h. 28 min. 40 sec. of 1 day. Hence,

If every fourth year is reckoned as leap year, 3 of every 4 centennial years excepted, the time *gained* in the calendar will be,

In 400 yr., 2 h. 28 min. 40 sec.
" 4000 " 24 " 46 " 40 "

The following rule for leap year will therefore render the calendar correct to within 1 day, for a period of 4000 years.

I. *Every year that is exactly divisible by* 4 *is a leap year, the centennial years excepted; the other years are common years.*

II. *Every centennial year that is exactly divisible by* 400 *is a leap year; the other centennial years are common years.*

NOTES. — 1. Julius Cæsar, the Roman Emperor, decreed that the year should consist of 365 days 6 hours; that the 6 hours should be disregarded for 3 successive years, and an entire day be added to every fourth year. This day was inserted in the calendar between the 24th and 25th days of February, and is called the *intercalary* day. As the Romans counted the days backward from the first day of the following month, the 24th of February was called by them *sexto calendas Martii*, the sixth before the calends of March. The intercalary day which followed this was called *bis-sexto calendas Martii;* hence the name *bissextile*.

2. In 1582 the error in the calendar as established by Julius Cæsar had increased to 10 days; that is, too much time had been reckoned as a year, until the civil year was 10 days behind the solar year. To correct this error, Pope Gregory decreed that 10 entire days should be stricken from the calendar, and that the day following the 3d day of October, 1582, should be the 14th. This brought the vernal equinox at March 21 — the date on which it occurred in the year 325, at the time of the Council of Nice.

3. The year as established by Julius Cæsar is sometimes called the *Julian year;* and the period of time in which it was in force, namely from 46 B.C. to 1582, A.D. is called the *Julian Period*.

4. The year as established by Pope Gregory is called the *Gregorian year*, and the calendar now used is the *Gregorian Calendar*.

5. Most Catholic countries adopted the Gregorian Calendar soon after it was established. Great Britain, however, continued to use the Julian Calendar until 1752. At this time the civil year was 11 days behind the solar year. To correct this error, the British Government decreed that 11 days should be stricken from the calendar, and that the day following the 2d day of September, 1752, should be the 14th.

6. Time before the adoption of the Gregorian Calendar is called *Old Style* (O.S.), and time since, *New Style* (N.S.). In Old Style the year commenced March 25, and in New Style it commences January 1.

7. Russia still reckons time by Old Style, or the Julian Calendar; hence the Russians' dates are now 12 days behind ours.

8. The centuries are numbered from the commencement of the Christian era; the months from the commencement of the year; the days from the commencement of the month, and the hours from the commencement of the day (12 o'clock, midnight). Thus, May 23, 1890, 9 o'clock A.M., is the 9th hour of the 23d day of the 5th month of the 90th year of the 19th century.

CIRCULAR OR ANGULAR MEASURE

378. Circular Measure, or Circular Motion, is used principally in surveying, navigation, astronomy, and geography, to measure arcs of angles or circles, for reckoning latitude and longitude, determining locations of places and vessels, and computing difference of time.

Every circle, great or small, is divisible into the same number of equal parts: as quarters, called **Quadrants**; twelfths, called **Signs**; 360ths, called **Degrees**, etc. Consequently the parts of different circles, although having the same names, are of different lengths.

The *unit* is the **Degree,** which is $\frac{1}{360}$ part of the space about *a point in* any plane. The table is made up of divisors and *multiples of this* unit.

TABLE.

60 seconds (″)	= 1 minute (′).
60 minutes	= 1 degree (°).
30 degrees	= 1 sign (S.).
12 signs, or 360°,	= 1 circle (C.).

UNIT EQUIVALENTS.

C.	S.	°	′	″
			1 =	60.
		1 =	60 =	3600.
	1 =	30 =	1800 =	108000.
1 =	12 =	360 =	21600 =	1296000.

SCALE — ascending, 60, 60, 30, 12; descending, 12, 30, 60, 60.

NOTES. — 1. Minutes of the earth's circumference are called geographic or nautical miles.

2. The denomination, *sign*, is confined exclusively to Astronomy.

3. A degree has no fixed linear extent. When applied to any circle it is always $\frac{1}{360}$ part of the circumference. But, strictly speaking, it is not any part of a circle.

4. 90° make a quadrant or rightangle; 60° make a sextant or $\frac{1}{6}$ of a circle.

MISCELLANEOUS TABLES.

379. COUNTING.

12 units = 1 dozen (doz.).
12 dozen = 1 gross (gro.).
12 gross = 1 great gross (G. gro.).
20 units = 1 score (sc.).

380. PAPER.

24 sheets = 1 quire (qre.).
20 quires = 1 ream (rm.).
2 reams = 1 bundle (bdl.).
5 bundles = 1 bale (B.).

381. BOOKS.

The terms *folio, quarto, octavo, duodecimo,* etc., indicate the number of leaves into which a sheet of paper is folded.

A sheet folded in 2 leaves is called a folio.
A sheet folded in 4 leaves " a quarto, or 4to.
A sheet folded in 8 leaves " an octavo, or 8vo.
A sheet folded in 12 leaves " a 12mo.
A sheet folded in 16 leaves " a 16mo.
A sheet folded in 18 leaves " an 18mo.
A sheet folded in 24 leaves " a 24mo.
A sheet folded in 32 leaves " a 32mo.

382. COPYING.

72 words make 1 folio or sheet of common law.
90 " " 1 " " " " chancery.

DENOMINATE NUMBERS.

383. A **Denominate Number** is a number composed of one or more units of any denomination, and may be *simple* or *compound.*

384. A **Simple Denominate Number** is a number composed of but one denomination, as, 7 miles; 6 hours; 5 tons.

385. A **Compound Denominate Number** is a number composed of two or more denominations of the same nature; as, 10 lb. 6 oz.; 6 yr. 3 mo.

386. A **Denominate Fraction** is a concrete fraction whose integral unit is *one* of a denomination of some compound number. Thus, $\frac{3}{7}$ of a day is a denominate fraction, the integral unit being one day; so are $\frac{5}{8}$ of a bushel, $\frac{2}{3}$ of a mile, etc., denominate fractions.

387. Most of the tables in Denominate Numbers, except those in the Metric System, are based on varying scales.

NOTE. — The only other tables that are not based on a varying scale are the table of counting 12 things make 1 dozen, 12 dozen 1 gross, 12 gross 1 great gross, etc., based on the duodecimal scale; the table of United States Currency, and some of the tables of Foreign Currency, based on the decimal scale.

388. **Reduction** is the process of changing a number from one denomination to another without altering its value. Reduction is of two kinds, Descending and Ascending.

389. **Reduction Descending** is changing a number of one denomination to a lower denomination or one of *less unit value;* thus, $1 = 10 dimes = 100 cents = 1000 mills.

390. **Reduction Ascending** is changing a number of one denomination to a higher denomination or one of *greater unit value;* thus, 1000 mills = 100 cents = 10 dimes = $1.

REDUCTION DESCENDING.

Examples.

391. To reduce a compound denominate number to lower denominations.

1. Reduce 3 mi. 57 rd. 2 yd. 1 ft. 8 in. to inches.

OPERATION.

3 mi. 57 rd. 2 yd. 1 ft. 8 in.
320
1017 rd.
$5\frac{1}{2}$
$508\frac{1}{2}$
5085
$5595\frac{1}{2}$ yd.
3
$16787\frac{1}{2}$ ft.
12
201458 in. *Ans.*

SOLUTION.—Since in 1 mile there are 320 rd., in 3 miles there are 3 × 320 rd. = 960 rd., and the 57 rd. in the given number added, = 1017 rd. in 3 mi. 57 rd. Since in 1 rd. there are $5\frac{1}{2}$ yd., in 1017 rd. there are 1017 × $5\frac{1}{2}$ yd. = $5593\frac{1}{2}$ yd., which with the 2 yd. in the given number = $5595\frac{1}{2}$ yd. in 3 mi. 57 rd. 2 yd. Since in 1 yd. there are 3 ft., in $5595\frac{1}{2}$ yd. there are $5595\frac{1}{2}$ × 3 ft. = $16786\frac{1}{2}$ ft., which with the 1 ft. in the given number = $16787\frac{1}{2}$ ft. in 3 mi. 57 rd. 2 yd. 1 ft. And since in 1 ft. there are 12 in., in $16787\frac{1}{2}$ ft. there are $16787\frac{1}{2}$ × 12 in. = 201450 in., which with the 8 in. in the given number = 201458 in. in the given compound number. On examining the operation, we find that we have successively multiplied by the numbers in the descending scale of linear measure from miles to inches, inclusive. As either factor may be used as a multiplicand, we may consider the numbers in the descending scale as multipliers.

RULE.—I. *Multiply the highest denomination of the given compound number by that number of the scale which will reduce it to the next lower denomination, and add to the product the given number, if any, of that lower denomination.*

II. *Proceed in the same manner with the results obtained in each lower denomination, until the reduction is brought to the denomination required.*

392. All the numbers in the metric system are based on a uniform scale of 10, 100, or 1000. Hence we have the following principle for the reduction of metric denominations.

PRINCIPLE.—*A denominate number in the metric system may be reduced to the next higher or lower denomination by the removal of the decimal point one place to the left or right in*

measures of length, capacity, and weight; two places to the left or right in measures of surface, and three places to the left or right in measures of volume, thus:

REDUCTION DESCENDING.

5.625^{Km}	$= 56.25^{Hm}$	$= 562.5^{Dm}$	$= 5625^{m}$	$= 56250^{dm}$
$2.325^{sq\ m}$	$= 232.5^{sq\ dm}$	$= 23250^{sq\ cm}$	$= 2325000^{sq\ mm}$	
$.532195^{cu\ m}$	$= 532.195^{cu\ dm}$	$= 532195^{cu\ cm}$	$= 532195000^{cu\ mm}$	

REDUCTION ASCENDING.

56250^{dm}	$= 5625^{m}$	$= 562.5^{Dm}$	$= 56.25^{Hm}$	$= 5.625^{Km}$
$2325000^{sq\ mm}$	$= 23250^{sq\ cm}$	$= 232.5^{sq\ dm}$	$= 2.325^{sq\ m}$	
$532195000^{cu\ mm}$	$= 532195^{cu\ cm}$	$= 532.195^{cu\ dm}$	$= .532195^{cu\ m}$	

NOTE. — When numbers in the metric system are reduced to like denominations, they may be added, subtracted, multiplied, or divided like any other whole numbers or decimals.

2. How many grains are there in 16 lb. 10 oz. 18 pwt. 5 gr.?

3. How many farthings are there in £133 6*s.* 8*d.*?

4. Change 100 mi. to inches.

5. How many rods of fence will inclose a farm 1½ miles square?

6. The gray limestone of Central New York weighs 175 lb. to the cubic foot. What is the weight of a block 8 ft. long and 1 yd. square?

7. What will be the cost of 1 hogshead of molasses at $.28 per gallon.?

8. Express 3.565^{g} in centigrams.

9. A man wishes to ship 1548 bu. 1 pk. of potatoes in barrels containing 2 bu. 3 pk. each. How many barrels must he obtain?

10. A grocer bought 10 bushels of chestnuts at $3.75 a bushel, and retailed them at $.06¼ a pint. What was his whole gain?

11. Reduce 90° 17′ 40″ to seconds.

12. How many days were there in the 18th century?

13. At 6¼ cts. each, what will be the cost of a great gross of writing books?

14. Reduce 6 O. 14 f℥ 3 fʒ 45 ♏ to minims.

15. How large an edition of an octavo book can be printed from 4 bales 4 bundles 1 ream 10 quires of paper, allowing 8 sheets to the volume?

16. Suppose your age to be 18 yr. 24 da. How many minutes old are you, allowing 4 leap years to have occurred in that time?

17. How many pence are there in 481 sovereigns?

18. Reduce \$7$\frac{3}{8}$ to mills.

19. In £6 10*s.* 10*d.* how many dollars United States currency are there?

20. Which is the cheaper and how much, to buy ribbon at \$.35 a yard or at \$.40 a meter?

21. How many dollars Canada currency are equal to £126 12*s.*?

22. How many square centimeters are there in a lot which is 50 ft. long and 100 ft. wide?

23. How many steps of 2 ft. 9 in. each will a man take in walking a distance of 95 miles?

24. How many decimeters are there in 5 yards?

25. Express .0037081 $^{\text{Kl}}$ in hectoliters; in centiliters.

26. A grocer bought 12 barrels of cider at \$1$\frac{3}{4}$ a barrel, and after converting it into vinegar, he retailed it at 6 cents a quart. What was his whole gain?

27. In 75 A. 4 sq. ch. 18 sq. rd. 118 sq. l. how many square links are there?

28. How many inches high is a horse that measures 16 hands?

29. If a vessel sails 150 leagues in a day, how many statute miles does she sail?

30. If 14 A. are sold from a field containing 50 A., how many square rods will the remainder contain?

31. A man returning from Pike's Peak has 36 lb. 8 oz. of pure gold; what is its value at \$1.04$\frac{1}{6}$ per pwt.?

32. I paid \$360 for 2 tons of cheese, and retailed it for 30 cents a kilogram. What was my whole gain?

33. A man having 8 hhd. of tobacco, each weighing 9 cwt. 42 lb., wishes to put it into boxes containing 48 lb. each. How many boxes must he obtain?

34. Write 56.232 kilometers as meters; as centimeters.

35. A merchant bought 12 barrels of salt at $\$1\frac{1}{4}$ a barrel, and retailed it at $\frac{3}{4}$ of a cent a pound. How much did he gain altogether?

36. A physician bought 1 ℔ 10 ℥ of quinine at \$2.25 an ounce, and dealt it out in doses of 10 gr. at $\$.12\frac{1}{2}$ a dose. How much more than cost did he receive?

37. Change 22350 sq. kilometers to sq. meters; to acres.

38. How many cubic centimeters are there in 2534 cubic meters?

393. To reduce a denominate fraction from a greater to a less unit.

1. Reduce $\frac{1}{44}$ of a gallon to the fraction of a gill.

OPERATION.

$$\frac{1}{\cancel{44}_{11}} \text{ gal.} \times \frac{\cancel{4}}{1} \times \frac{2}{1} \times \frac{4}{1} = \frac{8}{11} \text{ gi. } Ans.$$

SOLUTION.—To reduce gallons to gills, we multiply successively by 4, 2, and 4, the numbers in the descending scale. And since the given number is a fraction, we indicate the process, as in multiplication of fractions, after which we perform the indicated operations, and obtain $\frac{8}{11}$, the answer.

RULE.—*Multiply the fraction of the higher denomination by the numbers in the descending scale successively, between the given and the required denomination.*

NOTE.—Cancellation may be applied wherever practicable.

2. Reduce $\frac{1}{800}$ of a lb. Troy to the fraction of a pennyweight.

3. Reduce $\frac{1}{672}$ of a hhd. to the fraction of a pint.

4. Reduce $\frac{1}{2112}$ of a mile to the fraction of a yard.

5. Reduce $\frac{9}{842}$ of a gallon to the fraction of a gill.

6. What part of a dram is $\frac{1}{5000}$ of $\frac{2}{3}$ of $\frac{3}{4}$ of $\frac{6}{11}$ of $3\frac{1}{7}$ pounds, avoirdupois weight?

7. Reduce $\frac{1}{18000}$ of a dollar to the fraction of a cent.

8. Reduce $\frac{1}{45}$ of a rod to the fraction of a link.

9. Reduce $\frac{1}{85}$ of a scruple to the fraction of a grain.

10. What fraction of a yard is $\frac{6}{7}$ of $\frac{4}{11}$ of a rod?

11. $\frac{5}{18}$ of a week is $\frac{2}{9}$ of how many days?

12. What fraction of a square rod is $\frac{3}{1360}$ of $4\frac{1}{4}$ times $\frac{2}{19}$ of an acre?

394. To reduce a denominate fraction to integers of lower denominations.

1. What is the value of $\frac{2}{5}$ of a bushel?

OPERATION.

$\frac{2}{5}$ of 4 pk. $= \frac{8}{5}$ pk. $= 1\frac{3}{5}$ pk.
$\frac{3}{5}$ of 8 qt. $= \frac{24}{5}$ qt. $= 4\frac{4}{5}$ qt.
$\frac{4}{5}$ of 2 pt. $= \frac{8}{5}$ pt. $= 1\frac{3}{5}$ pt.
1 pk. 4 qt. $1\frac{3}{5}$ pt. *Ans.*

SOLUTION.—$\frac{2}{5}$ bu. $= \frac{2}{5}$ of 4 pk., or $1\frac{3}{5}$ pk.; $\frac{3}{5}$ pk. $= \frac{3}{5}$ of 8 qt. $= 4\frac{4}{5}$ qt.; and $\frac{4}{5}$ qt. $= \frac{4}{5}$ of 2 pt. $= 1\frac{3}{5}$ pt. The units, 1 pk., 4 qt., 1 pt., with the last denominate fraction, $\frac{3}{5}$ pt., form the answer.

RULE.—I. *Multiply the fraction by that number in the scale which will reduce it to the next lower denomination, and if the result is an improper fraction, reduce it to a whole or mixed number.*

II. *Proceed with the fractional part, if any, as before, until the number is reduced to the denominations required.*

III. *The units of the several denominations, arranged in their order, will be the required result.*

2. Reduce $\frac{9}{16}$ of a yard to integers of lower denominations.

3. Reduce $\frac{4}{5}$ of a month to lower denominations.

4. Reduce $\frac{607}{640}$ of a short ton to lower denominations.

5. What is the value of $\frac{5}{9}$ of a long ton?

6. What is the value of $\frac{3}{8}$ of $2\frac{1}{2}$ pounds apothecaries' weight?

7. What is the value of $\frac{7}{13}$ of an acre?

8. Reduce $\frac{3}{7}$ of a mile to integers of lower denominations.

9. What is the value of $\frac{4}{7}$ of a great gross?

10. What is the value in geographic miles of $\frac{9}{16}$ of a great circle?

11. What is the value of $\frac{4}{5}$ of $3\frac{3}{8}$ cords of wood?

12. The distance between A and B is $1\frac{1}{2}$ miles; having traveled $\frac{2}{5}$ of this distance from A, how much further must I travel to reach B?

13. What is the value of $\frac{43}{96}$ f℥ ?

14. What is the value of $\frac{3}{7}$ of a sign?

15. A man having a hogshead of molasses, sold $\frac{6}{13}$ of it. How much remained?

395. To reduce a denominate decimal to integers of lower denominations.

1. Reduce .125 of a barrel to integers of lower denominations.

OPERATION.

```
  .125
  31.5
 ------
 3.9375 gal.
      4
 ------
 3.7500 qt.
     2
 ------
  1.50 pt.
    4
 ------
   2.0 gi.
```

3 gal. 3 qt. 1 pt. 2 gi. *Ans.*

SOLUTION. — We first multiply the given decimal, .125 of a barrel, by 31.5 to reduce it to gallons, and obtain 3.9375 gallons. Omitting the 3 gallons, we multiply the decimal, .9375 gal., by 4 to reduce it to quarts, and obtain 3.75 quarts. We next multiply the decimal part of this result by 2, to reduce it to pints, and obtain 1.5 pints. The decimal part of this result we multiply by 4 to reduce it to gills, and obtain 2 gills. The integers of the several denominations, arranged in their order, form the answer.

RULE. — I. *Multiply the given denominate decimal by that number in the descending scale which will reduce it to the next lower denomination and point off the result as in multiplication of decimals.*

II. *Proceed with the decimal part of the product in the same manner until reduced to the required denominations. The integers of the several denominations will form the answer required.*

Reduce to lower denominations:

2. .645 day.	**5.** .875 hhd.	**8.** .625 fath.
3. .765 lb. Troy.	**6.** £.85251.	**9.** .8469°.
4. .6625 mi.	**7.** .715°.	**10.** .375 yd.

11. How many centiliters are 3.756 hectoliters?

REDUCTION ASCENDING.

Examples.

396. To reduce a compound denominate number to higher denominations.

1. Reduce 157540 minutes to weeks.

OPERATION.

$$\begin{array}{r l}
60\underline{)157540} & \text{min.} \\
24\underline{)2625} & \text{h.} + 40 \text{ min.} \\
7\underline{)109} & \text{da.} + 9 \text{ h.} \\
15 & \text{wk.} + 4 \text{ da.}
\end{array}$$

15 wk. 4 da. 9 h. 40 min. *Ans.*

SOLUTION. — Dividing the given number of minutes by 60, because there are $\frac{1}{60}$ as many hours as minutes, we obtain 2625 h. plus a remainder of 40 min. We next divide the 2625 h. by 24, because there are $\frac{1}{24}$ as many days as hours, and we find that 2625 h. = 109 da. plus a remainder of 9 h. Lastly we divide the 109 da. by 7, because there are $\frac{1}{7}$ as many weeks as days, and we find that 109 da. = 15 wk. plus a remainder of 4 da.

2. Reduce 201458 inches to miles.

OPERATION.

$$\begin{array}{r l}
12\underline{)201458} & \text{in.} \\
3\underline{)16788} & \text{ft. 2 in.} \\
5\tfrac{1}{2} \text{ or } 5.5\underline{)5596} & \text{yd.} \\
40\underline{)\ 1017} & \text{rd. 2 yd. 1 ft. 6 in.} \\
8\underline{)25} & \text{fur. 17 rd.} \\
3 & \text{mi. 1 fur.}
\end{array}$$

3 mi. 1 fur. 17 rd. 2 yd. 1 ft. 8 in. *Ans.*

SOLUTION. — We divide successively by the numbers in the ascending scale of linear measure, in the same manner as in the last example. But, in dividing 55960 yd. by $5\frac{1}{2}$ or 5.5, we have a remainder of $2\frac{1}{2}$ yd., and this reduced to its equivalent compound number (§ 394) = 2 yd. 1 ft. 6 in. In forming our final result, the 6 in. of this number is added to the first remainder, 2 in., making 8 in.

RULE. — I. *Divide the given concrete or denominate number by that number of the ascending scale which will reduce it to the next higher denomination.*

II. *Divide the quotient by the next higher number in the scale; and so proceed to the highest denomination required. The last quotient, with the several remainders annexed in a reversed order, will be the answer.*

NOTE. — The several corresponding cases in reduction descending and reduction ascending, being opposite, naturally prove each other.

3. Reduce 1913551 drams to tons.

4. In 97920 gr. of medicine how many lb. are there?

5. Reduce 1000000 in. to mi.

6. How many acres are there in a field 120 rd. long and 56 rd. wide?

7. How many cords are there in a pile of wood 60 ft. long, 15 ft. wide, and 10 ft. high?

8. In 30876 gi. how many hhd. are there?

9. How many bushels of corn are 27072 qt.?

10. How many hectograms are there in 565 grams.

11. In 1234567 far. how many £ are there?

12. Reduce 2468 pence to half crowns.

13. In 84621 ♏ how many Cong. are there?

14. At $4 a hectoliter, what is the value of 4 liters of apples?

15. If 135 million Spencerian steel pens are manufactured yearly, how many great gross will be made each year?

16. Reduce 1020300″ to S.

17. In 411405 seconds how many days are there?

18. During a storm at sea, a ship changed her latitude 412 geographic miles. How many degrees and minutes did she change?

19. In 120 gross how many score are there?

20. How many miles are there in the semicircumference of the earth?

21. How much time will a person gain in 36 yr. by rising 45 min. earlier, and retiring 25 min. later, every day, allowing for 9 leap years?

22. If I walk at the rate of 264 feet in a minute, how many miles an hour do I walk?

23. How many acres are there in a field 290 rd. long and 96 rd. wide?

24. How many steres are there in a pile of wood 20^{m} long, 5^{m} wide, and 3^{m} high?

25. Express 7645^{cl} as a decimal of a hectoliter.

26. How many steres are there in 256 decisteres?

27. A coal dealer bought 175 long tons of coal at $3.75, and sold it at $4.50 a short ton. What was his whole gain?

28. An Ohio farmer sold a load of corn weighing 2912 lb., and a load of wheat weighing 2400 lb.; for the corn he received $.60 a bushel, and for the wheat $1.20 a bushel. How much did he receive for both loads?

29. Reduce 25 cubic centimeters to cubic inches.

30. An English grocer sold 120 barrels of apples, each containing 2 bu. 2 pk., at 4*s.* 7*d.* a bushel, and received pay in cloth at 10*s.* 5*d.* a yard. How many yards of cloth did he receive?

OPERATION.

$$\frac{120 \times \overset{.1}{\cancel{2.5}} \times \overset{11}{\cancel{55}}}{\cancel{125}} = 132 \text{ yd. } \textit{Ans.}$$

SOLUTION. — The operation in this example is similar to the preceding examples, except that we divide the *cost* of the apples by the *price of a unit* of the article received in payment, reduced to units of the same denomination as the price of a *unit* article sold. Thus 4*s.* 7*d.* = 55*d.*, the cost per bushel, and 120 bbl. of apples each containing 2½ bu. will cost 120 × 2.5 × 55*d.* 10*s.* 5*d.* = 125*d.*, the cost of 1 yd.; hence $\frac{120 \times 2.5 \times 55}{125}$ = the number of yards. *Ans.*, 132.

31. If I make a box with a bottom 4 ft. square, how high must I make it to have it contain 160½ cu. ft.?

32. If a cistern has 14⅞ sq. ft. on the bottom, how deep must it be to contain 940 cu. ft.? How many gallons of water will it contain?

33. A tank 7⅓ ft. by 5¼ ft. by 12 ft. will contain how many barrels of kerosene?

34. How many bushels of wheat are there in a freight car 32 ft. long, 8 ft. wide, the wheat being 2 ft. 2 in. deep in the car?

35. A man bought 200 gallons of astral oil at $.06 a gallon, and sold it at $.03 a liter. How much did he gain?

36. How many loads of earth must be removed to grade down a street 1 ft. 6 in., the street being ½ mi. in length and 20 ft. wide?

37. How many loads of earth will be required to raise the grade of a lot 2 ft., the lot being 55 ft. by 135 ft.?

38. What is the cost of 20 yd. 2 ft. 6 in. of wire at 6 cents per yard?

39. What is the difference in size between a garden containing 6 sq. rd. and one which is 6 rods square?

40. A barrel contains 28 gal. of oysters; after selling 8 gal. 2 qt. 1 pt., how many gallons are left?

41. In a pile of wood 60 ft. long, 20 ft. wide, and 15 ft. high, how many cords are there?

42. How many barrels, each containing 2 bu. 2 pk., will be required to hold 81 bushels of apples?

43. At the rate of $\frac{3}{4}$ of a mile per minute, how far will a train run in $3\frac{1}{4}$ hours?

44. How many Troy pounds are there in 3730.373 kilograms?

45. What part of a square yard is a strip of cloth 20 in. by 15 in.?

46. If a load of wood is 12 ft. long and 3 ft. wide, how high must it be to make a cord?

47. Mt. Everest is 29062 ft. high. How many miles high is it?

48. A pile of wood 9 ft. long, $7\frac{1}{2}$ ft. high, and 5 ft. thick, costs $9. What is the price per cord?

49. Anthracite coal weighs 93.75 lb. per cubic foot. How many cubic feet do 2062.5 lb. occupy?

50. How much will it cost to fill a bin 315 ft. by 4.5 ft. by 6.5 ft., with coal 40 cu. ft. to the ton, at $7.48 per ton?

51. What is the cost of 17 meters of lace at 8 cents per decimeter?

52. A bin is 2 meters by 2 meters by 6 meters. How many hectoliters of wheat will it contain?

53. How many steres are there in a pile of wood 20^m long, 1.25^m wide, and 4^m high?

54. How many cords of wood are there in 10 steres?

397. To reduce a denominate fraction from a less to a greater unit.

1. Reduce $\frac{8}{11}$ of a gill to the fraction of a gallon.

OPERATION.

$$\frac{8}{11}\text{ gi.} \times \frac{1}{4} \times \frac{1}{2} \times \frac{1}{4} = \frac{1}{44}\text{ gal. } \textit{Ans.}$$

SOLUTION. — To reduce gills to gallons, we divide successively by 4, 2, and 4, the numbers in the ascending scale. And since the given number is a fraction, we indicate the process, as in division of fractions, after which we perform the indicated operations, and obtain $\frac{1}{44}$, the answer.

RULE. — *Divide the fraction of the lower denomination by the numbers in the ascending scale successively, between the given and the required denomination.*

NOTE. — The operation may frequently be shortened by cancellation.

2. Reduce $\frac{2}{3}$ of a shilling to the fraction of a pound.

3. Reduce $\frac{5}{7}$ of a pennyweight to the fraction of a pound Troy weight.

4. What part of a ton is $\frac{4}{5}$ of a pound avoirdupois weight?

5. What fraction of an hour is $\frac{4}{9}$ of 20 seconds?

6. What part an hour is $\frac{3}{4}$ of 28 minutes?

7. $\frac{3}{4}$ of $\frac{5}{8}$ of a rod is what part of a mile?

8. Change $\frac{2}{3}$ of a dozen to the fraction of a gross?

9. What is the fractional difference between $\frac{1}{680}$ of a hhd. and $\frac{2}{3}$ of a pt.?

10. $\frac{224}{3}$ of $\frac{1}{8}$ of $\frac{2}{7}$ of a pint is what fraction of 2 pecks?

11. Reduce $\frac{2}{3}$ of $\frac{4}{7}$ of $\frac{9}{14}$ of $\frac{7}{8}$ of a cord foot to the fraction of a cord.

12. What part of an acre is $\frac{3}{19}$ of $\frac{4}{17}$ of $9\frac{1}{2}$ square rods?

13. $\frac{3}{4}$ of $5\frac{1}{2}$ furlongs is $\frac{1}{2}$ of $\frac{1}{12}$ of how many miles?

14. A block of granite containing $\frac{3}{4}$ of $\frac{6}{7}$ of $20\frac{1}{8}$ cubic feet, is what fraction of a perch?

15. What part of a cord of wood is a pile $7\frac{1}{2}$ ft. long, 2 ft. high, and $3\frac{1}{4}$ feet wide?

16. Reduce $\frac{5}{7}$ of an inch to the fraction of an English ell.

398. To reduce a compound denominate number to a fraction of a higher denomination.

1. Reduce 2 oz., 12 pwt. 12 gr. to the fraction of a pound Troy.

OPERATION.

2 oz. 12 pwt. 12 gr. = 1260 gr.
1 lb. Troy = 5760 gr.
$\frac{1260}{5760}$ lb. = $\frac{7}{32}$ lb. *Ans.*

SOLUTION. — To find what part one compound number is of another, they must be like numbers and reduced to the same denomination. In 2 oz. 12 pwt. 12 gr. there are 1260 gr., and in 1 lb. there are 5760 gr. Therefore 1 gr. is $\frac{1}{5760}$ lb., and 1260 gr. are $\frac{1260}{5760}$ lb. = $\frac{7}{32}$ lb.

RULE. — *Reduce the given number to its lowest denomination for the numerator, and a unit of the required denomination to the same denomination for the denominator of the required fraction.*

NOTE. — If the given number contains a fraction, the denominator of this fraction must be regarded as the lowest denomination.

2. Reduce 60 sq. rd. to the fraction of an acre.

3. What part of a mile is 6 fur. 26 rd. 3 yd. 2 ft.?

4. What part of a £ is 18*s.* 5*d.* $2\frac{2}{13}$ far.?

5. What part of 21 ℔. apothecaries' weight is 7 ℥ 7 ʒ 2 ℈ 14 gr.?

6. What part of 3 weeks is 4 da. 16 h. 30 min.?

7. Reduce $1\frac{3}{4}$ pecks to the fraction of a bushel.

8. From a hogshead of molasses 28 gal. 2 qt. were drawn. What part of the whole remained in the hogshead?

9. Reduce 4 bundles 6 quires 16 sheets of paper to the fraction of a bale.

10. What part of 54 cords of wood is 4800 cubic feet?

11. What is the value of $\dfrac{7\frac{4}{11}}{1\frac{18}{77}}$ of a dollar?

12. Reduce 3 O. 3 f ℥ 1 f ʒ 36 ♏ to the fraction of a Cong.

13. What part of a ton of hewn timber is 36 cu. ft. 864 cu. in.?

14. Reduce 1 month 3 days 5 hours 15 minutes to a fraction of a year.

399. To reduce a compound denominate number to a decimal of a higher denomination.

1. Reduce 3 cd. ft. 8 cu. ft. to the decimal of a cord.

OPERATION.

$$\begin{array}{r|l} 16 & 8.0 \text{ cu. ft.} \\ \hline 8 & 3.5000 \text{ cd. ft.} \\ \hline & .4375 \text{ Cd. } Ans. \end{array}$$

Or,

3 cd. ft. 8 cu. ft. = 56 cu. ft.

1 Cd. = 128 cu. ft.

$\frac{56}{128}$ Cd. = $\frac{7}{16}$ Cd. = .4375 Cd. *Ans.*

SOLUTION. — We reduce the 8 cu. ft. to the decimal of a cd. ft., by annexing a cipher, and dividing by 16, the number of cu. ft. in 1 cd. ft.; then we annex the decimal quotient to the 3 cd. ft. We now reduce the 3.5 cd. ft. to Cd. or a decimal of a Cd., by dividing by 8, the number of cd. ft. in 1 Cd., and we have .4375 Cd., the answer.

Or, we may reduce the 3 cd. ft. 8 cu. ft. to the fraction of a Cd. (as in § 398), and we shall have $\frac{56}{128}$ Cd. = $\frac{7}{16}$ Cd., which reduced to its equivalent decimal, equals .4375 Cd., the same as before.

RULE. — *Divide the lowest denomination given, by that number in the scale which will reduce it to the next higher denomination, and annex the quotient as a decimal to that higher denomination. Proceed in the same manner until the whole is reduced to the denomination required.* Or,

Reduce the given number to a fraction of the required denomination, and reduce this fraction to a decimal.

2. Reduce 5 da. 9 h. 46 min. 48 sec. to the decimal of a week.

3. Reduce 3° 27′ 46.44″ to the decimal of a sign.

4. What part of 4 oz. is 2 oz. 16 pwt. 19.2 gr.?

5. What part of a furlong is 28 rd. 2 yd. 1 ft. 11.04 in.?

6. Reduce $3\frac{1}{4}$ ℥ to the decimal of a pound.

7. Reduce 126 A. 4 sq. ch. 12 sq. rd. to the decimal of a township.

8. What part of a fathom is $3\frac{3}{4}$ ft.?

9. What part of $1\frac{1}{4}$ bushels is .45 of a peck?

10. Reduce $\frac{2}{7}$ of $\frac{1}{2}$ of $22\frac{3}{4}$ lb. to the decimal of a short ton.

11. What part of a f℥ is 5 fʒ 36 ♏?

12. Reduce 50 gal. 3 qt. 1 pt. to the decimal of a tun.

13. Express 2015 hectoliters in deciliters.

ADDITION.

400. Compound denominate numbers are added, subtracted, multiplied, and divided by methods based on the same principles as simple numbers; and the only modification of the operations and rules is that required for borrowing, carrying, and reducing by a *varying* instead of a *uniform scale.*

Examples.

401. To add compound denominate numbers.

1. What is the sum of 50 hhd. 32 gal. 3 qt. 1 pt.; 2 hhd. 19 gal. 1 pt.; 15 hhd. $46\frac{1}{4}$ gal.; and 9 hhd. 39 gal. $2\frac{1}{2}$ qt.

hhd.	gal.	qt.	pt.
50	32	3	1
2	19	0	1
15	46	1	0
9	39	2	1
78	11	3	1

SOLUTION. — Writing the numbers so that units of the same denomination stand in the same column, we add the numbers of the right hand or lowest denomination, and find the amount to be 3 pints, which is equal to 1 qt. 1 pt. We write the 1 pt. under the column of pints, and add the 1 qt. to the column of quarts. The amount of the numbers of the next higher denomination is 7 qt., which is equal to 1 gal. 3 qt. We write the 3 qt. under the column of quarts, and add the 1 gal. to the column of gallons. Adding the gallons, we find the amount to be 137 gal., equal to 2 hhd. 11 gal. Writing the 11 gal. under the gallons in the given numbers, we add the 2 hhd. to the column of hogsheads. Adding the hogsheads, we find the amount to be 78 hhd., which we write under the left-hand denomination, as in simple numbers.

2. What is the sum of $\frac{7}{10}$ wk., $\frac{3}{5}$ da., and $\frac{3}{8}$ h.?

OPERATION.

	da.	h.	min.	sec.
$\frac{7}{10}$ wk. =	4 da.	21 h.	36 min.	
$\frac{3}{5}$ da. =		14 "	24 "	
$\frac{3}{8}$ h. =			22 "	30 sec.
	5	12	22	30 *Ans.*

Or,

$\frac{3}{5}$ da $\times \frac{1}{7} = \frac{3}{35}$ wk.;

$\frac{3}{8}$ h. $\times \frac{1}{24} \times \frac{1}{7} = \frac{1}{448}$ wk.;

$\frac{7}{10}$ wk. + $\frac{3}{35}$ wk. + $\frac{1}{448}$ wk. $= \frac{1765}{2240}$ wk.

$= \frac{353}{448}$ wk.;

$\frac{353}{448}$ wk. = 5 da. 12 h. 22 min. 30 sec. *Ans.*

SOLUTION. — We first find the value of each fraction in integers of lower denominations (§ 394), and then add the results.

Or, we may reduce the given fractions to fractions of the same denomination (§ 393 or 397), then add them, and find the value of their sum in lower denominations.

RULE. — I. *If any of the numbers are denominate fractions, or if any of the denominations are mixed numbers, reduce the fractions to integers of lower denominations.*

II. *Write the numbers so that those of the same unit value stand in the same column.*

III. *Beginning at the right hand, add each denomination as in simple numbers, carrying to each succeeding denomination one for as many units as it takes of the denomination added, to make one of the next higher denomination.*

NOTE. — The pupil cannot fail to see that the principles involved in adding compound numbers are the same as those in addition or simple numbers; and that the *only difference* consists in the different carrying units.

3.

lb.	oz.	pwt.	gr.
14	6	12	13
17	5	3	12
15		9	16
2	7	15	20
13	2	1	19
4	1	5	21

4.

℔.	℥	ʒ	℈	gr.
10	8	5	1	8
7	7	6	2	13
5	11	7		
21	10			16
12	1	2	2	3
		7	1	19

5.

fur.	rd.	ft.	in.
7	26	11	9
4	16	7	11
	36	14	3
1	9	2	8
5		10	1
6	2	5	
1	15	13	10

6.

A.	sq. rd.	sq. yd.	sq. ft.
140	137	27	6
320	70	14	2
111		3	
214	95	22	7
100	120	6	1
25	76		8
104	89	1	4

7. Add 1 T. 17 cwt. 8 lb., 5 cwt. 29 lb. 8 oz., 1 cwt. 42 lb. 6 oz., and 17 lb. 8 oz.

8. Add 6 yd. 2 ft., 3 yd. 1 ft. 8 in., 1 ft. $10\frac{1}{2}$ in., 2 yd. 2 ft. $6\frac{1}{2}$ in., 2 ft. 7 in., and 2 yd. 5 in.

9. Add 4 Cd. 7 cd. ft., 2 Cd. 2 cd. ft. 12 cu. ft., 6 cd. ft. 15 cu. ft., 5 Cd. 3 cd. ft. 8 cu. ft., and 2 Cd. 1 cu. ft.

10. Find the sum of 12.65^{m}, 2000^{mm}, 3.456^{cm}, and 15^{Km}.

11. What is the sum of $1\frac{2}{3}$ hhd. 42 gal. 3 qt. $1\frac{1}{4}$ pt., $\frac{7}{8}$ gal. 2 qt. $\frac{3}{4}$ pt., and 1.75 pt.?

12. What is the sum of $145\frac{7}{8}$ A., 7 A. $109\frac{1}{2}$ sq. rd., 1 A. 136.5 sq. rd., and $\frac{5}{6}$ A.?

13. What is the sum of 31 bu. 2 pk., $10\frac{7}{8}$ bu., 5 bu. $6\frac{1}{2}$ qt., 14 bu. 2.75 pk., and $\frac{2}{3}$ pk.

14. Add 54.3^{l}, 2340^{cl}, and 519^{Hl}.

15. Add 3 ft. 5 in., 2 yd. 1 ft. 1 in., and 6 yd. 8 in.

16. Add 42 yr. $7\frac{1}{2}$ mo., 10 yr. 3 wk. 5 da., $9\frac{3}{4}$ mo. 1 wk. 16 h. 40 min., $\frac{5}{6}$ mo. $3\frac{3}{4}$ da.

17. Add 3 S. 22° 50′, 24° 36′ 25.7″, 17′ 18.2″, 1 S. 3° 12′ 15.5″, 12° 36′ 17.8″, and 57.3″.

18. Find the sum of $25^{sq\,m}$, $3560^{sq\,cm}$, $248^{sq\,Dm}$, $20000^{sq\,mm}$, and $23478^{sq\,dm}$.

19. How many units are $1\frac{1}{2}$ gross $7\frac{1}{3}$ doz., 3 gross $1\frac{3}{4}$ doz., $\frac{3}{4}$ of a great gross, $6\frac{1}{4}$ doz., and 4 doz. 7 units?

20. If I buy the N. E. $\frac{1}{4}$ and the E. $\frac{1}{2}$ of N. W. $\frac{1}{4}$ of a section of land, how many acres do I purchase? How are the parts located in respect to each other?

21. Add $3\frac{3}{4}$ yd., 5 ft. 6 in., and $1\frac{1}{8}$ yd.

22. What is the sum of 3 ℔ 5 ℥ 4 ʒ 2 ℈ 17 gr., 2 ℔ 5 ʒ 12 gr., 4 ℥ 2 ʒ 1 ℈ 16 gr.?

23. A New York farmer received \$.60 a bushel for 4 loads of corn; the first contained 42.4 bu., the second 2866 lb., the third $36\frac{3}{4}$ bu., and the fourth 39 bu. 29 lb. How much did he receive for the whole?

24. Three loads of hay were bought at \$8 per ton. The first weighed 1.125 T., the second $1\frac{2}{5}$ T., and the third 2500 pounds; how much did the whole cost?

25. What is the distance in kilometers around a lot which is 100 ft. long and 50 ft. wide?

26. A man in digging a cellar removed $140\frac{4}{5}$ cu. yd. of earth, in digging a cistern 24.875 cu. yd., and in digging a drain 46 cu. yd. $20\frac{1}{4}$ cu. ft. What was the amount of earth removed, and what was the cost at 18¢ a cu. yd.?

SUBTRACTION.

Examples.

402. To subtract compound denominate numbers.

1. From 18 lb. 5 oz. 4 pwt. 14 gr. take 10 lb. 6 oz. 10 pwt. 8 gr.

OPERATION.

lb.	oz.	pwt.	gr.
18	5	4	14
10	6	10	8
7	10	14	6 *Ans.*

SOLUTION. — Writing the subtrahend under the minuend, placing units of the same denomination under each other, we subtract 8 gr. from 14 gr. and write the remainder, 6 gr., underneath. Since we cannot subtract 10 pwt. from 4 pwt., we add 1 oz. or 20 pwt. to the 4 pwt., subtract 10 pwt. from the sum, and write the remainder, 14 pwt., underneath. Having added 20 pwt. or 1 oz. to the 6 oz. in the subtrahend, we find that we cannot subtract the sum, 7 oz., from the 5 oz. in the minuend; we therefore add 1 lb. or 12 oz. to the 5 oz., subtract 7 oz. from the sum, and write the remainder, 10 oz., underneath. Adding 12 oz. or 1 lb. to the 10 lb. in the subtrahend, we subtract the sum, 11 lb., from the 18 lb. in the minuend, as in simple numbers, and write the remainder, 7 lb., underneath.

2. From 12 bbl. 15 gal. 3 qt. take 7 bbl. 18 gal. 1 qt.

OPERATION.

bbl.	gal.	qt.
12	15	3
7	18	1
4	$28\frac{1}{2}$	2
4	29	*Ans.*

SOLUTION. — Proceeding as in the last operation, we obtain a remainder of 4 bbl. $28\frac{1}{2}$ gal. 2 qt. But $\frac{1}{2}$ gal. = 2 qt., which added to the 2 qt. in the remainder = 1 gal., and this added to the 28 gal. = 29 gal.; and the answer is 4 bbl. 29 gal.

3. From $\frac{3}{4}$ of a rod subtract $\frac{3}{4}$ of a yard.

OPERATION.

	yd.	ft.	in.
$\frac{3}{4}$ rd. =	4	0	$4\frac{1}{2}$
$\frac{3}{4}$ yd. =		2	3
	3	1	$1\frac{1}{2}$ *Ans.*

Or,

$\frac{3}{4}$ yd. $\times \frac{1}{5\frac{1}{2}} = \frac{3}{4}$ yd. $\times \frac{2}{11} = \frac{3}{22}$ rd.;

$\frac{3}{4}$ rd. $- \frac{3}{22}$ rd. $= \frac{27}{44}$ rd.;

$\frac{27}{44}$ rd. = 3 yd. 1 ft. $1\frac{1}{2}$ in. *Ans.*

SOLUTION. — We first find the value of each fraction in integers of lower denominations (§ 394), and then subtract the less value from the greater. Or, we may reduce the given fractions to fractions of the same denomination, subtract the less value from the greater, and find the value of the remainder in integers of lower denominations.

*Rule. — I. *If any of the numbers are denominate fractions, or if any of the denominations are mixed numbers, reduce the fractions to integers of lower denominations.*

II. *Write the subtrahend under the minuend, so that units of the same denomination stand under each other.*

III. *Beginning at the right hand, subtract each denomination separately, as in simple numbers.*

IV. *If the number of any denomination in the subtrahend exceeds that of the same denomination in the minuend, add to the number in the minuend as many units as make one of the next higher denomination, and then subtract; in this case add 1 to the next higher denomination of the subtrahend before subtracting, or subtract 1 from the next higher denomination of the minuend. Proceed in the same manner with each denomination.*

4.

	mi.	fur.	rd.	ft.	in.
From	175	3	27	11	4
Take	59	6	10	12	9

5.

A.	sq. rd.	sq. yd.
320	3	26.4
150	2	31.86

6.

	hhd.	gal.	qt.
From	5	36	$3\frac{1}{4}$
Take	2	45	$1\frac{7}{8}$

7.

yr.	mo.	wk.	da.	h.
[illegible]	1	3	0	$17\frac{1}{4}$
10	9	1	22	6.8

8. Subtract 15 rd. 10 ft. $3\frac{1}{4}$ in. from 26 rd. 11 ft. 3 in.

9. From 1 T. 11 cwt. 30 lb. 6 oz. take 18 cwt. 45 lb.

10. Subtract .659 wk. from 2 wk. $3\frac{5}{8}$ da.

11. From $\frac{117}{224}$ hhd. take .90625 gal.

12. From $\frac{3}{8}$ of $3\frac{3}{4}$ A. take 3 sq. rd.

13. Subtract $\frac{3}{40}$ lb. Troy from 10 lb. 8 oz. 8 pwt.

14. From a pile of wood containing 36 Cd. 4 cd. ft. there was sold 10 Cd. 6 cd. ft. 12 cu. ft. How much remained?

15. From $5\frac{1}{2}$ bbl. take $\frac{4}{7}$ of a hogshead.

16. Subtract $\frac{327}{360}$ of a day from $\frac{3}{8}$ of a week.

17. From $\frac{5}{8}$ of a gross subtract $\frac{3}{8}$ of a dozen.

18. From $\frac{7}{8}$ of a mile take $\frac{21}{22}$ of a rod.

19. Subtract 2 A. 3 sq. rd. 5.76 sq. yd. from 5 A. 1 sq. rd. 24.24 sq. yd.

20. If a speculator buys the N. half of a section of land and sells at one time the N.W. $\frac{1}{4}$ of N.W. $\frac{1}{4}$, at another time the N.E. $\frac{1}{2}$ of N.W. $\frac{1}{4}$, and at another time the N.E. $\frac{1}{4}$ of N.E. $\frac{1}{4}$, how many acres will he have? Draw a diagram showing their location.

21. Subtract .0625 bu. from $\frac{3}{4}$ pk.

22. From the sum of $\frac{5}{8}$ of $365\frac{1}{4}$ days and $\frac{3}{4}$ of $5\frac{5}{8}$ weeks take $49\frac{1}{7}$ minutes.

23. From the sum of $\frac{2}{7}$ of $3\frac{3}{4}$ mi. and $17\frac{1}{7}$ rd. take $5\frac{1}{3}$ fur.

24. From a piece of cloth containing 36.57 meters, 12 yards were cut; how many yards remained? How many meters remained?

25. From 15 bbl. 3.25 gal. take 14 bbl. 24 gal. 3.54 qt.

26. A farmer in Ohio having 200 bu. of barley, sold 3 loads, the first weighing 1457 lb., the second 1578 lb., and the third 1420 lb. How many bushels had he left?

27. From 25.8^{Kg} take 326^{g}. How much remains?

28. Of a farm containing 200 acres two lots were reserved, one containing 50 A. 136.4 sq. rd., and the other 48 A. 123.3 sq. rd.; the remainder was sold at $35 per acre. How much was received for it?

29. An excavation 58 ft. long, 37 ft. wide, and 6 ft. deep is to be made for a cellar. After 471 cu. yd. 16 cu. ft. 972 cu. in. of earth have been removed, how much still remains to be taken out?

30. From a piece of cloth containing 45.75^{m} a tailor cut at different times 5 suits of clothes, each containing 7.5^{m}. How much remained?

31. From the sum of $\frac{4}{5}$ lb., $4\frac{5}{6}$ oz., and $31\frac{1}{3}$ pwt., take the difference between $\frac{3}{5}$ oz. and $\frac{7}{8}$ pwt.

32. From the sum of $5\frac{7}{12}$ A., $\frac{3}{8}$ of $6\frac{1}{4}$ A., and $\frac{3}{11}$ of $2\frac{2}{11}$ sq. rd., take 4 A. 25 sq. rd. 12 sq. yd.

403. To find the difference in dates.

1. How many years, months, days, and hours is it from 3 o'clock P.M. of June 15, 1887, to 10 o'clock A.M. of Feb. 22, 1895?

OPERATION.

yr.	mo.	da.	h.	
1895	2	22	10	
1887	6	15	15	
7	8	6	19	*Ans.*

SOLUTION. — Since the later of two dates always expresses the greater period of time, we write the later date for a minuend and the earlier date for a subtrahend, placing the denominations in the order of the descending scale from left to right. 10 o'clock A.M., Feb. 22, 1895, is equivalent to 1895 yr. 2 mo. 22 da. 10 h., and 3 P.M., June 15, 1887 = 1887 yr. 6 mo. 15 da. and 12 + 3 or 15 h. We then subtract by the rule for subtraction of compound numbers.

RULE. — *Write the later date as a minuend and the earlier date as a subtrahend, and subtract.*

When the *exact number of days* is required for any period not exceeding one ordinary year, it may be readily found by the following table.

TABLE.

Showing the number of days from any day of one month to the same day of any other month within one year.

From Any Day of	To the Same Day of the Next											
	Jan.	Feb.	Mar.	Apr.	May	June	July	Aug.	Sept.	Oct.	Nov.	Dec.
January . . .	365	31	59	90	120	151	181	212	243	273	304	334
February . .	334	365	28	59	89	120	150	181	212	242	273	303
March	306	337	365	31	61	92	122	153	184	214	245	275
April	275	306	335	365	30	61	91	122	153	183	214	244
May	245	276	304	335	365	31	61	92	123	153	184	214
June	214	245	273	304	334	365	30	61	92	122	153	183
July.	184	215	243	274	304	335	365	31	62	92	123	153
August. . . .	153	184	212	243	273	304	334	365	31	61	92	122
September. .	122	153	181	212	242	273	303	334	365	30	61	91
October . . .	92	123	151	182	212	243	273	304	335	365	31	61
November . .	61	92	120	151	181	212	242	273	304	334	365	30
December . .	31	62	90	121	151	182	212	243	274	304	335	365

If the days of the different months are not the same, the number of days of difference should be *added* when the earlier day belongs to the month *from* which we reckon, and *subtracted* when it belongs to the month *to* which we find the time. If the 29th of February is to be included in the time computed, one day must be added to the result.

2. The American Revolutionary War began April 19, 1775, and peace was restored Jan. 20, 1783; how long did the war continue?

3. How much time has elapsed since the Declaration of Independence of the United States?

4. The Pilgrims landed at Plymouth Dec. 22, 1620, and General Washington was born Feb. 22, 1732; what was the difference in time between these two events?

5. The first settlement made in the United States was at Jamestown, Va., May 23, 1607; how many years was it from that time to July 4, 1896?

6. How many days is it from the 6th of November to the 15th of the following January?

7. How many days is it, in leap year, from the 20th of August to the 15th of the following June?

8. What length of time elapsed from the death of Henry Wadsworth Longfellow, March 24, 1882, to that of James Russell Lowell, Aug. 12, 1891?

9. How long has a note to run, dated Jan. 30, 1893, and made payable June 3, 1895?

10. How long was it from the battle of Bunker Hill, June 17, 1775, to the battle of Waterloo, June 18, 1815?

11. How many years, months, and days is it from your birthday to this date?

12. What length of time elapsed from 16 minutes past 10 o'clock, A.M., July 4, 1890, to 22 minutes before 8 o'clock, P.M., Dec. 12, 1894?

13. What length of time will elapse from 3.45 P.M. of Feb. 22, 1896, to the end of the nineteenth century?

14. How many days is it from the 4th of September, 1895, to *the 27th of May* following?

MULTIPLICATION.

Examples.

404. To multiply compound denominate numbers.

1. Multiply 5 mi. 4 fur. 18 rd. 15 ft. by 6.

OPERATION.

5 mi.	4 fur.	18 rd.	15 ft.	
			6	
33	2	33	$7\frac{1}{2}$	*Ans.*

SOLUTION. — Writing the multiplier under the lowest denomination of the multiplicand, we multiply each denomination in the multiplicand separately, in order, from lowest to highest, as in simple numbers, and carry from lower denominations to higher, according to the ascending scale of the multiplicand, as in addition of compound numbers. Thus, 6 × 15 ft. = 90 ft., or 5 rd. and $7\frac{1}{2}$ ft.; 6 × 18 rd. + 5 rd. = 113 rd. = 2 fur. and 33 rd.; 6 × 4 fur. + 2 fur. = 26 fur. = 3 mi. and 2 fur.; 6 × 5 mi. + 3 mi. = 33 mi. *Ans.* 33 mi. 2 fur. 33 rd. $7\frac{1}{2}$ ft.

RULE. — *Write the multiplier under the lowest denomination of the multiplicand. Multiply as in simple numbers, and carry as in addition of compound numbers.*

NOTES. — 1. When the multiplier is large, and is a *composite* number, we may shorten the work by multiplying by the component factors. The multiplier must be an abstract number.

2. If any of the denominations are mixed numbers, they may either be reduced to integers of lower denominations before multiplying, or they may be multiplied as directed in § 262.

3. The multiplication of a denominate fraction is the most readily performed by § 262, after which the product may be reduced to integers of lower denominations by § 394.

As the work of multiplying by large prime numbers is somewhat tedious, the following method may often be so modified and adapted as to greatly shorten the operation.

2. How many bushels of grain are there in 47 bags, each containing 2 bu. 1 pk. 4 qt.?

FIRST OPERATION.

47 = (5 × 9) + 2.

2 bu.	1 pk.	4 qt.	
		5	
11 bu.	3 pk.	4 qt.	in 5 bags.
		9	
106 bu.	3 pk.	4 qt.	in 45 bags.
4 "	3 "		" 2 "
111 bu.	2 pk.	4 qt.	" 47 " *Ans.*

SOLUTION. — Multiplying the contents of 1 bag by 5, and the resulting product by 9, we have the contents of 45 bags, which is the composite number *next less* than the given prime number, 47. Next, multiplying the contents of 1 bag by 2, we have the contents of 2 bags, which, added to the contents of 45 bags, gives us the contents of 45 + 2 = 47 bags.

SECOND OPERATION.

$47 = (6 \times 8) - 1.$		
2 bu. 1 pk. 4 qt.		
6		
14 bu. 1 pk.	in	6 bags.
8		
114 bu.	in	48 bags.
2 " 1 pk. 4 qt.	"	1 bag.
111 bu. 2 pk. 4 qt.	"	47 bags. *Ans.*

Or, we may multiply the contents of 1 bag by the factors of the composite number *next greater* than the given prime number, 47, and from the last product subtract the multiplicand.

3.

	T.	cwt.	lb.	oz.
	12	15	27	9
				8
Prod.				

4.

mi.	fur.	rd.	ft.
14	6	36	14
			9

5.

	A.	sq. rd.	sq. yd.	sq. ft.
	7	73	21	7
				6
Prod.				

6.

Cd.	cd. ft.	cu. ft.
10	7	13
		12

7. Multiply 34 bu. 3 pk. 6 qt. 1 pt. by 14.

8. Multiply 4 lb. 10 oz. 18.7 pwt. by 27.

9. Multiply 9 ℥ 3 ʒ 2 ℈ 13 gr. by 35.

10. Multiply 5 gal. 2 qt. 1 pt. 3.25 gi. by 96.

11. Multiply 78 A. 15 sq. yd. by 15⅞.

12. What is 73 times 9 cu. yd. 10 cu. ft. 1424 cu. in.?

13. Multiply 2 lb. 8 oz. 13 pwt. 18 gr. by 59.

14. Multiply 4 yd. 1 ft. 4.7 in. by 125.

15. If 1 qt. 2 gi. of milk will fill 1 bottle, how much will be required to fill a gross of bottles of the same capacity?

16. How much will 120 yd. of cloth cost, at £1 9*s.* per yard?

17. What is the weight of 48 loads of hay, each weighing 1 T. 3 cwt. 50 lb.?

18. A milliliter of water weighs 1 gram; what is the weight in kilograms of 25^{m} of water?

19. Multiply 7 O. 10 f℥ 4 fʒ 25 ♏ by 24.

20. Multiply 3 hhd. 43 gal. 2.6 gi. by 17.

21. Multiply 9 T. 13 cwt. 1 qr. 10.5 lb. by 1.7.

NOTE. — When the multiplier contains a decimal, the multiplicand may be reduced to the lowest denomination mentioned, or the lower denominations to a decimal of the higher, before multiplying. The result can be reduced to the compound number required.

22. If a pipe discharges 2 hhd. 23 gal. 2 qt. 1 pt. of water in 1 hour, how much will it discharge in 4.8 hours, if the water flows with the same velocity?

23. What number of rails will inclose a quarter-section of land with a fence 6 rails high, and 3 lengths for every 2 rods? and what will be the cost of the rails at $40 per thousand?

24. What will be the value of 1 dozen gold cups, each cup weighing 9 oz. 13 pwt. 8 gr., at $212.38 a pound?

25. How many steres of wood are there in a pile 16^m long, 19^m wide, and 25^m high?

26. A farmer sold 5 loads of oats, averaging 37 bu. 3 pk. 5 qt. each, at $.65 per bushel. How much money did he receive for the grain?

27. How many acres are there in a field 57^m long and 28.5^m wide?

28. A speculator bought of the Ill. Central R. R. Co. the S. $\frac{1}{2}$ of Section 4, township 10 north, range 6 east, at $2 an acre. He sold the E. $\frac{1}{2}$ of the S.E. $\frac{1}{4}$ at $2.75 an acre; the N.W. $\frac{1}{4}$ of the S.E. $\frac{1}{4}$ at $$3\frac{1}{2}$ an acre, and the N. $\frac{1}{2}$ of the S.W. $\frac{1}{4}$ at $3.84 an acre. How many acres had he left? What was his gain on the purchase price of the whole? Draw diagram.

29. A man having purchased a section of land from the U. S. Government at $1.25 an acre, sold the S. $\frac{1}{2}$ of S.W. $\frac{1}{4}$ at $2.50 an acre, the N.W. $\frac{1}{4}$ of N.W. $\frac{1}{4}$ at $1.75 an acre, the W. $\frac{1}{2}$ of S.E. $\frac{1}{4}$ at $2 an acre, and the W. $\frac{1}{2}$ of S.W. $\frac{1}{4}$ of N.E. $\frac{1}{4}$ at $3 an acre. How many acres has he remaining, and what is his gain, provided the remainder is sold at $$2\frac{1}{4}$ an acre? Draw diagram and explain.

30. In 6 barrels of grain, each containing 2 bu. 3 pk. 5 qt., how many bushels are there?

DIVISION.

Examples.

405. To divide compound denominate numbers.

1. Divide 37 A. 60 sq. rd. 7 sq. yd. by 8.

OPERATION.

8)	37 A.	60 sq. rd.	7 sq. yd.	
	4	107	16	*Ans.*

SOLUTION. — Writing the divisor on the left of the dividend, we divide the highest denomination, and the quotient is 4 A. and the remainder 5 A. We write the quotient under the denomination divided, and reduce the remainder to sq. rd., making 800 sq. rd., which, added to the 60 sq. rd. of the dividend, equals 860 sq. rd. Dividing this, we have a quotient of 107 sq. rd., and a remainder of 4 sq. rd. Proceeding in the same way with the yards, our answer is 4 A. 107 sq. rd. 16 sq. yd.

2. Divide 111 bu. 2 pk. 4 qt. by 47.

OPERATION.

```
47)111 bu. 2 pk. 4 qt.(2 bu. 1 pk. 4 qt. Ans.
    94
    ---
    17 bu. rem.
     4
    ---
    70 pk. in 17 bu. 2 pk.
    47
    ---
    23 pk. rem.
     8
    ---
   188 qt. in 23 pk. 4 qt.
   188
   ---
```

SOLUTION. — The divisor being large, and a prime number, we divide by long division, setting down the whole work of subtracting and reducing.

RULE. — I. *Divide the highest denomination as in simple numbers, and each succeeding denomination in the same manner, if there is no remainder.*

II. *If there is a remainder after dividing any denomination, reduce it to the next lower denomination, adding in the given number of that denomination, if any, and divide as before.*

III. *The several partial quotients will be the quotient required.*

NOTE. — When the divisor is large and is a *composite* number, we may shorten the work by dividing by the factors. When the divisor and dividend are both compound numbers, they must both be reduced to the same denomination before dividing, and then the process is the same as in simple numbers. The division of a denominate fraction is most readily performed by §§ 263 and 394.

3.					4.			
	hhd.	gal.	qt.	pt.	T.	cwt.	qr.	lb.
12)	9	28	2	0	19) 373	19	2	4

5. Divide 358 A. 57 sq. rd. 6 sq. yd. 2 sq. ft. by 7.

6. Divide 192 bu. 3 pk. 1 qt. 1 pt. by 9.

7. How many hectoliters of grain can be put into a bin 3^m long, 2.5^m wide, and 1.75^m deep? How many bushels would such a bin hold?

8. Divide 336 yd. 4 ft. $3\frac{1}{8}$ in. by 21.

9. Divide 77 sq. yd. 5 sq. ft. 82 sq. in. by 13.

10. Divide 678 cu. yd. 1 cu. ft. 1038.05 cu. in. by 67.

11. Divide 1986 mi. 3 fur. 20 rd. 1 yd. by 108.

12. Divide 12 sq. mi. 30 sq. rd. by $22\frac{1}{2}$.

NOTE. — Observe that $22\frac{1}{2} = \frac{45}{2}$; hence, multiply by 2, and divide the result by 45.

13. Divide 365 da. 6 h. by 240.

14. A box 3.52^m long and 2.5^m wide contains $27.28^{cu\,m}$. How deep is it?

15. Divide 3794 cu. yd. 20 cu. ft. $709\frac{1}{3}$ cu. in. by $33\frac{1}{3}$.

16. Divide 121 lb. 3 ℥ 2 ʒ 1 ℈ 4 gr. by $13\frac{3}{4}$.

17. Divide 28° 51′ 27.765″ by 2.754.

18. Divide 202 yd. 1 ft. $6\frac{3}{4}$ in. by $\frac{3}{5}$.

19. Divide 1950 da. 15 h. $15\frac{5}{9}$ min. by 100.

20. If a town 4 miles square is divided equally into 124 farms, how much will each farm contain?

21. A pile of wood containing 36.25^{st} is 2^m long and 1.5^m wide. How high is it? How many cords does it contain?

22. A cellar 48 ft. 6 in. long, 24 ft. wide, and $6\frac{1}{2}$ ft. deep, was excavated by 6 men in 8 days; how many cubic yards did each man excavate daily?

23. How many casks, each containing 2 bu. 3 pk. 6 qt., can be filled from 356 bu. 3 pk. 5 qt. of cherries?

24. If a township of land is equally divided among 288 families, how many acres does each receive? What part of a section?

LONGITUDE AND TIME.

406. Every circle is supposed to be divided into 360 equal parts, called *degrees*. Since the sun appears to pass from east to west around the earth, or through 360°, once in every 24 hours, it will pass through $\frac{1}{24}$ of 360°, or 15° of the distance, in 1 hour; and 1° of distance in $\frac{1}{15}$ of 1 hour, or 4 minutes; and 1′ of distance in $\frac{1}{60}$ of 4 minutes, or 4 seconds, etc.

TABLE OF LONGITUDE AND TIME.

360° of longitude	= 24 hours of time.	15″ of longitude	= 1 second of time.
15° " "	= 1 hour "	1° " "	= 4 minutes "
15′ " "	= 1 minute "	1′ " "	= 4 seconds "

Hence, the difference in longitude in degrees, minutes, and seconds is 15 times the difference in time in hours, minutes, and seconds, respectively.

Examples.

407. To find the difference of longitude between two places, when the difference of time is known.

1. If the difference of time between New York and Cincinnati is 41 min. 36 sec., what is the difference of longitude?

FIRST OPERATION.

min.	sec.
41	36
	15
624′	0″ =
10°	24′ *Ans.*

SECOND OPERATION.

min.	sec.
4)41	36
10°	24′ *Ans.*

SOLUTION. — Since there are 15 times as many degrees, minutes, and seconds of longitude as there are hours, minutes, and seconds of time, we multiply the time by 15.

Or, since 4 minutes of time make a difference of 1° of longitude, and 4 seconds of time a difference of 1′ of longitude, there will be $\frac{1}{4}$ as many degrees of longitude as there are minutes of time, and $\frac{1}{4}$ as many minutes of longitude as there are seconds of time.

RULE. — *Multiply the difference in hours, minutes, and seconds of time by* 15; *the product will be the difference in longitude in degrees, minutes, and seconds.* Or,

Reduce the difference of time to minutes and seconds, and then divide by 4; *the quotient will be the difference in longitude, in degrees and minutes.*

NOTES. — 1. When as in the second operation we change minutes to degrees, and seconds to minutes, we multiply by 60; but since we also divided by 4, and $60 \div 4 = 15$, the result is the same as though we multiplied by 15, as in the first operation.

2. If one place is in east and the other in west longitude, the difference of longitude is found by *adding* them, and if the sum is greater than 180°, it must be subtracted from 360°.

3. Since the sun appears to move from east to west, when it is exactly 12 o'clock at one place, it will be *past* 12 o'clock at all places east, and *before* 12 at all places west. Hence, if the difference of time between two places is *subtracted* from the time at the *easterly* place, the result will be the time at the westerly place; and if the difference is *added* to the time at the *westerly* place, the result will be the time at the easterly place.

2. When it is 9 o'clock at Washington, it is 8 h. 7 min. 12 sec. at St. Louis; the longitude of Washington being 77° 3′ west, what is the longitude of St. Louis?

3. The sun rises at Boston 1 h. 16 min. 4 sec. sooner than at New Orleans; the longitude of New Orleans being 90° 5′ west, what is the longitude of Boston?

4. When it is half past 2 o'clock in the morning at Havana, it is 9 h. 13 min. 24 sec. A.M. at the Cape of Good Hope; the longitude of the latter place being 18° 29′ east, what is the longitude of Havana?

5. The difference of time between Valparaiso and Rome is 5 h. 36 min. 36 sec. What is the difference in longitude?

6. When it is 12 o'clock M. at San Francisco it is 2 h. 58 min. 13 sec. P.M. at Rochester, N.Y.; the longitude of the latter place being 77° 51′ W., what is the longitude of San Francisco?

7. A man traveling West from Quebec, which is in 71° 13′ 45″ W. longitude, finds, on his arrival at Denver, Colo., that his watch is 1 h. 38 min. 49 sec. faster than true time at the latter place. If his watch has kept accurate time, what is the longitude of Denver?

8. A ship's chronometer, set at Greenwich, points to 5 h. 40 min. 20 sec. P.M., when the sun is on the meridian. What is the ship's longitude?

NOTE. — Greenwich, Eng., is on the meridian of 0°, and from this meridian longitude is reckoned.

9. The longitude of Stockholm is 18° 3′ 45″ E.; when it is midnight there, it is 6 h. 8 min. 15.3 sec. P.M. at New York. What is the longitude of New York from Greenwich?

408. To find the difference of time between two places, when their longitudes are given.

1. The longitude of Boston is 71° 4′, and of Chicago 87° 37′. What is the difference of time between these two places.

FIRST OPERATION.

$$\begin{array}{r} 87^\circ\ 37' \\ \underline{71^\circ\ \ 4'} \\ 15\overline{)16^\circ\ 33'} \text{ difference of long.} \\ 1 \text{ h. } 6 \text{ min. } 12 \text{ sec. } \textit{Ans.} \end{array}$$

SECOND OPERATION.

$$\begin{array}{l} 16^\circ \quad 33' \\ \underline{\qquad\quad 4} \\ 64 \text{ min. } 132 \text{ sec.} = \\ 1 \text{ h. } 6 \text{ min. } 12 \text{ sec. } \textit{Ans.} \end{array}$$

SOLUTION.—By subtraction of compound numbers we first find the difference of longitude between the two places, which is 16° 33′ Since the difference in longitude in degrees, minutes, and seconds, is 15 times the difference of time in hours, minutes, and seconds, $\frac{1}{15}$ of 16° 33′ = 1 h. 6 min. 12 sec., the difference in time.

Or, since 1° of longitude makes a difference of 4 minutes of time, and 1′ of longitude a difference of 4 seconds of time, we multiply 16° 33′, the difference in longitude, by 4, and we obtain the difference of time in minutes and seconds, which, reduced to higher denominations, gives 1 h. 6 min. 12 sec., the difference in time.

RULE.—*Divide the difference of longitude in degrees, minutes, and seconds, by* 15, *and the quotient will be the difference of time in hours, minutes, and seconds.* Or,

Multiply the difference of longitude in degrees and minutes by 4, *and the product will be the difference of time in minutes and seconds, which may be reduced to hours.*

NOTE.—When as in the second operation we change degrees to minutes, and minutes to seconds, we divide by 60, but since we also multiply by 4, the result is the same as when we divide by 15 in the first operation.

2. New York is 74°, and Cincinnati 84° 26′ west longitude. What is the difference of time?

3. The Cape of Good Hope is 18° 29′ east, and the Sandwich Islands 155° west longitude. What is the difference of time?

4. If a message is sent by telegraph without any loss of time, at 12 M. from London, 0° 0′ longitude, to Washington, 77° 3′ west, what is the time of its receipt at Washington?

5. Washington is 77° 3′ west, and St. Petersburg 30° 18′ east longitude. What is their difference of time?

6. A steamer arrives at Halifax, 63° 35′ west, at 4 o'clock P.M.; the fact is telegraphed to St. Louis, 90° 15′ west, without loss of time. What is the time of its receipt at St. Louis?

7. If Pekin is 116° 28′ 54″ east, and San Francisco 122° 24′ 15″ west longitude, what is their difference of time?

8. If at a presidential election, the voting begins at sunrise and ends at sunset, how much sooner will the poles open and close at Portland, Me., 70° 15′ 40″ west, than at Portland, Oregon, 122° 27′ 30″ west?

9. When it was 1 o'clock A.M., on the first day of January, 1891, at Bangor, Me., 68° 47′ west, what was the time at the city of Mexico, 99° 5′ west?

10. A steamer arrives at Halifax, 63° 35′ west, at 4 h. 30 min. P.M.; the fact is telegraphed to New York, 74° west, without loss of time. What is the time of its receipt at New York?

409. To avoid the inconvenience arising from the difference in time between places comparatively near (as, for example, the difference of 12 minutes between New York and Boston), the following standard time divisions have been adopted by the railroads, and local time is now generally superseded by standard time.

410. **Standard Time Divisions as adopted by the Railroads.** — EASTERN STANDARD. — 75th meridian. Canada, between Quebec and Detroit; United States east of Buffalo, New York; Pittsburg, Pennsylvania; Wheeling and Huntington, West Virginia; Bristol, Tennessee; Charlotte, North Carolina, and Augusta, Georgia.

CENTRAL STANDARD. — 90th meridian. West from "Eastern" limits, as above, to Broadview, Canada; to the Missouri River in Dakota; North Platte and McCook, Nebraska; Wallace and Dodge City, Kansas; Toyah and Sanderson, Texas.

MOUNTAIN STANDARD. — 105th meridian. West from "Central" limits to Heron, Montana; Ogden, Utah; Needles and Yuma, Arizona.

PACIFIC STANDARD. — 120th meridian. West from "Mountain" limits to coast.

411. As a matter of fact, standard time coincides with local time in very few places in the United States, and therefore for accurate calculations the correction is constantly needed.

Eastern Standard Time—that of the 75th meridian—coincides very nearly with local time at Potsdam and Herkimer, New York; Camden and Cape May, New Jersey, and Philadelphia, Pennsylvania.

Central Standard Time—that of the 90th meridian—is practically local time for Bessemer, Michigan; Mt. Carroll, Illinois; St. Louis, Missouri; Memphis, Tennessee; Jackson, Mississippi; and New Orleans, Louisiana.

Mountain Time is very nearly local time at Cheyenne, Wyoming; Denver, Colorado; and Las Vegas, New Mexico.

Pacific Time is very close to local time at Reno and Carson City, Nevada, and Santa Barbara, California.

While the standard meridians are exactly 15° apart, the belts using the several meridian times vary, according to convenience, from 10° to 25° in width. This results in some singular variations. For instance, El Paso, Texas, has the same standard time as Savannah, Georgia, though there is a difference of 1 h. 45 min. in local time. On the other hand, San Diego, California, has two hours later standard time than El Paso, although there is a difference of but 42 min. in local time.

By knowing the meridian, which is the standard for any place, and the distance of the meridian from the place in degrees east or west, local time can be converted into standard time.

For a place *east* of its standard meridian, the time of the rising or setting of any heavenly body may be expressed in standard time by subtracting from the almanac calendar time one minute of time for every 15′ of longitude, or 4 minutes of time for every degree of longitude that the place is east from the standard meridian.

For a place *west* of its standard meridian, the time of the rising or setting of the sun or moon may be found by adding to the almanac calendar time one minute of time for every 15′ of longitude, or 4 minutes of time for every degree of longitude that the place is west of the standard meridian.

412. Corrections for the following Cities.

Eastern Standard. 75° Longitude.	Min.	Central Standard. 90° Longitude.	Min.	Mountain Standard. 105° Longitude.	Min.
Bangor, Me.	−25	Cleveland, Ohio	−33	Denver Col.	...
Augusta, Me.	−21	Columbus, Ohio	−28	Salt Lake City, Utah	+28
Portland, Me.	−19	Detroit, Mich.	−28		
Boston, Mass.	−16	Toledo, Ohio	−26	**Pacific Standard. 120° Longitude.**	
Newport, R.I.	−15	Dayton, Ohio	−23		
Providence, R.I.	−14	Cincinnati, Ohio	−22	Sacramento, Cal.	+ 6
Concord, N.H.	−14	Louisville, Ky.	−18	San Francisco, Cal.	+10

1. Buffalo is about 79° west longitude. What is the difference between standard time and actual time in that city?

2. Cincinnati is 84° 26′ west longitude. When it is noon by actual time, what is the hour by standard time?

3. Washington is 77° 3′ west longitude. When it is noon by standard time, what is the actual time?

4. What is the longitude of Bangor? Of Newport? Of Louisville? Of Detroit? Of San Francisco?

NOTE. — Refer to table of corrections.

5. When the sun rises in Boston at 6 o'clock in the morning, as indicated by the almanac, at what hour does it rise by standard time?

6. The longitude of Springfield, Mass., is 72° 35′ 45″ W., and of Galveston, Tex., 94° 50′ W.; when it is 20 min. past 6 o'clock A.M. at Springfield, what time is it at Galveston? What is the difference between the actual time and standard time?

7. The longitude of Boston, Mass., is 71° 4′ W., and of San Francisco, Cal., 122° 24′ W. When it is noon at Boston, what time is it at San Francisco? What is the difference between the actual time and standard time?

413. Miscellaneous Examples in Denominate Numbers.

1. How much will 3 cwt. 12 lb. of hay cost at $15½ a ton?

2. How many grains are 9 lb. 8 ℥ 1 ʒ 2 ℈ 19 gr.?

3. How many English ells are there in 27 yd. 2 qr.?

4. Reduce $18.945 to sterling money.

5. In 4 yr. 48 da. 10 h. 45 sec. how many seconds are there?

6. How many printed pages, 2 pages to each leaf, will there be in an octavo book having 24 fully printed sheets?

7. At 1/6 sterling per yard, how many yards of cloth may be bought for £ 5 6*s.* 6*d.*?

8. In 4 mi. 51 ch. 73 l. how many links are there?

9. In 22 A. 153 sq. rd. 2¾ sq. yd. how many square yards are there?

10. How many demijohns, each containing 3 gal. 1 qt. 1 pt., can be filled from 3 hhd. of vinegar?

11. A man sent a silver tray and pitcher, weighing 3 lb. 9 oz., to a jeweler, and ordered them made into teaspoons, each weighing 1 oz. 5 pwt.; how many spoons ought he to receive, making no allowance for waste?

12. What part of 4 gal. 3 qt. is 2 qt. 1 pt. 2 gi.?

13. Reduce $\frac{3}{7}$ of $\frac{4}{11}$ of a rod to the fraction of a yard.

14. What must be the depth of a bin 1^m wide and 2^m deep to contain 6200^l of grain?

15. How many yards of carpeting 1 yd. wide will be required to cover a floor 26½ ft. long and 20 ft. wide?

16. If I purchase 15 T. 3 cwt. 3 qr. 24 lb. of English iron, by long ton weight, at 6 cents a pound, and sell the same at $140 per short ton, how much do I gain by the transaction?

17. What will be the expense of plastering a room 40 ft. long, 36½ ft. wide, and 22¼ ft. high, at 18 cents a sq. yd., allowing 1375 sq. ft. for doors, windows, and baseboard?

18. How much tea is there in 23 chests, each weighing 78 lb. 9 oz.?

19. Valparaiso is in latitude 33° 2′ south, and Mobile 30° 41′ north; what is their difference of latitude?

20. If a druggist sells 1 gross 4 dozen bottles of Congress water a day, how many will he sell during the month of July?

21. If I use 15 kilograms of coal a day, how many tons will I use at that rate in a year?

22. Eighteen buildings are erected on an acre of ground, each occupying, on an average, 4 sq. rd. 120 sq. ft. 84 sq. in.; how much ground remains unoccupied?

23. At $13 per ton, how much hay may be bought for $12.02½?

24. If 1 pk. 4 qt. wheat cost $.33, how much will a bushel cost?

25. Find the degree of the angle which the hour and minute hands of a clock form at 5 o'clock.

SOLUTION.—The minute hand will be at 12. Since the hour hand travels 360° in twelve hours, it travels 30° every hour, hence at 5 o'clock it will make an angle of 5 × 30° with the minute hand.

26. Find the degree of the angle which the hour and minute hands make at 2 o'clock. At 3, 4, 6, 7, 8, 9, 10, 11.

27. How many bushels are there in 36000 lb. of wheat?

28. At what times between 4 and 5 o'clock will the hour and minute hands of a clock form a straight line? A right angle? An angle of 45°?

29. At 20 cents a cubic yard, how much will it cost to dig a cellar 32 ft. long, 24 ft. wide, and 6 ft. deep?

30. If the wall of the same cellar is laid 1½ feet thick, what will it cost at $1.25 a perch?

31. The forward wheels of a wagon are 10 ft. 4 in. in circumference, and the hind wheels 15½ ft.; how many more times will the forward wheels revolve than the hind wheels in running from Boston to New York, the distance being 248 miles?

32. A man bought 15 cwt. 22 lb. of rice at $3.75 a cwt., and 7 cwt. 36 lb. of pearl barley at $4.25 a cwt. How much would he gain by selling the whole at 4½ cents a pound?

33. If a barometer shows the pressure of air at a certain place to be 34 in., to what point would the mercury rise in another barometer which registers in millimeters at the same place?

34. From $\frac{5}{7}$ of 3 T. 10 cwt. subtract $\frac{4}{13}$ of 7 T. 3 cwt. 26 lb.

35. What is the value in avoirdupois weight of 16 lb. 5 oz. 10 pwt. 13 gr. Troy?

36. What decimal of a rod is 1 ft. 7.8 in.?

37. If a piece of timber is 9 in. wide and 6 in. thick, what length of it will be required to make 3 cu. ft.?

38. If a board is 16 in. broad, what length of it will make 7 sq. ft.?

39. How many tons are there in a stick of hewn timber 60 ft. long, and 1 ft. 9 in. by 1 ft. 1 in.?

40. Subtract $\dfrac{7\frac{1}{2}}{8\frac{1}{8}}$ bu. + $\frac{5}{8}$ of $\frac{56}{75}$ of $3\frac{1}{8}$ qt. from 5 bu. $3\frac{21}{45}$ qt.

41. How many pounds of silver, Troy weight, are equivalent in value to 5.6 lb. of gold by the English government standard?

42. A grocer bought 275 bu. of corn at $.75 a bushel and sold it at $3 a hectoliter. How much did he gain?

43. If a piece of gold is $\frac{5}{8}$ pure, how many carats fine is it?

44. In gold 16 carats fine what part is pure and what part is alloy?

45. A man having a piece of land containing $384\frac{5}{8}$ A., divided it between his two sons, giving to the elder 22 A. 60 sq. rd. more than to the younger. How many acres did he give to each?

46. If a boy walks 80^m a minute, how many miles an hour will he walk at the same rate?

47. The great pyramid of Cheops measures 763.4 feet on each side of its base, which is square. How many acres does it cover?

48. The roof of a house is 42 ft. long, and each side 20 ft. 6 in. wide. What will the roofing cost at 4.62\frac{1}{2}$ a square?

49. If a man buys 10 hectoliters of chestnuts at $13 a hectoliter, and sells them at $5.50 a bushel, how much does he gain?

50. If 17 T. 15 cwt. 62½ lb. of iron cost $1333.593, how much will 1 ton cost?

51. A cubic foot of distilled water weighs 1000 ounces avoirdupois. What is the weight of a liquid gallon?

52. There is a house 45 feet long, and each of the two sides of the roof is 22 feet wide. Allowing each shingle to be 4 inches wide and 15 inches long, and to lie one third to the weather, how many half thousand bunches will be required to cover the roof?

53. A cistern measures 4 ft. 6 in. square, and 6 ft. deep; how many hogsheads of water will it hold?

54. If the driving wheels of a locomotive are 18 ft. 9 in. in circumference, and make 3 revolutions in a second, how long will the locomotive be in running 150 miles?

55. In traveling, when I arrived at Louisville my watch, which was exactly right at the beginning of my journey, and a correct timekeeper, was 1 h. 6 min. 52 sec. fast. From what direction had I come and how far?

56. How many U.S. bushels will a bin contain that is 8.5 ft. long, 4.25 ft. wide, and 3¾ ft. deep?

57. Reduce 3 hhd. 9 gal. 3 qt. liquid measure to Imperial gallons.

58. A man owns a piece of land which is 105 ch. 85 l. long, and 40 ch. 15 l. wide; how many acres does it contain?

59. A and B own a farm together; A owns $\frac{7}{12}$ of it and B the remainder, and the difference between their shares is 15 A. 68½ sq. rd. What is B's share?

60. At $3.40 per square, what will be the cost of tinning both sides of a roof 40 ft. in length, whose rafters are 20 ft. 6 in. long?

61. If a horse eats 2.5^{Dl} of oats in a week, how long will 12 bushels last him?

[redacted]

62. What is the value of a farm 189.5 rd. long and 150 rd. wide, at \$ $31\frac{3}{4}$ per acre?

63. Reduce 9.75 tons of hewn timber to feet, board measure; that is, 1 in. thick.

64. How many gallons will a tank contain that is 4 ft. long, $3\frac{1}{4}$ ft. wide, and $2\frac{6}{7}$ ft. deep?

65. If a load of wood is 12 ft. long, and 3 ft. 6 in. wide, how high must it be to make a cord?

66. The scale of a map of New York State is 1^{cm} to 5^{Km}. The distance between two places on this map measures 12^{cm}. What is their actual distance in miles?

67. In a schoolroom 32 ft. long, 18 ft. wide, and 12 ft. 6 in. high, there are 60 pupils, each breathing 10 cu. ft. of air in a minute. In how long a time will they breathe as much air as the room contains?

68. A man has a piece of land $201\frac{3}{8}$ rods long and $41\frac{1}{4}$ rods wide, which he wishes to lay out into square lots of the greatest possible size. How many lots will there be?

69. A man has 4 pieces of land, containing 4 A. 140 sq. rd., 6 A. 132 sq. rd., 9 A. 120 sq. rd., and 11 A. 112 sq. rd. respectively. It is required to divide each piece into the largest sized building lots possible, each lot containing the same area, and an exact number of square rods. How much land will each lot contain?

70. How many perches of masonry are there in the wall of a cellar which is 20 ft. square on the inside, 8 ft. high, and $1\frac{1}{2}$ ft. in thickness?

71. A vessel set sail from New York, and proceeded in a southeasterly direction for 24 days. The captain then took an observation of the sun, and found the local time at the ship's meridian to be 10 h. 4 min. 36.8 sec. A.M. At the moment of the observation, his chronometer, which had been set for New York time, showed 8 h. 53 min. 47 sec. A.M. Allowing that the chronometer had gained 3.56 sec. per day, how much had the ship changed her longitude since she set sail?

PRACTICAL MEASUREMENTS.

LUMBER.

BOARDS, TIMBER, ETC.

414. The *unit* of measure for boards, timber, plank, and joist is a square foot, 1 inch in thickness, which is called a **Board Foot.**

In **Board Measure** all boards are assumed to be 1 inch thick unless otherwise specified. All boards over 1 inch in thickness to $1\frac{1}{4}$ inches are reckoned as $1\frac{1}{4}$ inches; over $1\frac{1}{4}$ to $1\frac{1}{2}$ inches as $1\frac{1}{2}$ inches; $1\frac{1}{2}$ to 2 inches as 2 inches; over 2 inches according to their thickness. Thus a board $1\frac{1}{4}$ inches thick is equal to $1\frac{1}{4}$ boards of the same size 1 inch thick; and a board 2 inches thick is equal to 2 boards of the same size 1 inch thick.

In practice the width of a board is reckoned only to the next smaller half-inch. Thus a width of $5\frac{3}{8}$ is reckoned as 5 inches; a width of $5\frac{3}{4}$ is reckoned as $5\frac{1}{2}$ inches.

415. Since a board foot is only $\frac{1}{12}$ of a foot thick, it contains $1 \times 1 \times \frac{1}{12} = \frac{1}{12}$ cu. ft.; hence it takes 12 board feet to make a cubic foot. Board feet are therefore changed to cubic feet by dividing by 12, and cubic feet are changed to board feet by multiplying by 12.

416. Lumber and **Sawed Timber**, as plank, scantling, etc., are usually estimated in *board* measure, **Hewn** and **Round Timber** in *cubic* measure.

Examples.

417. To find the square contents of boards, joists, etc.

1. Find the contents of a board 15 ft. long, 8 in. wide.

OPERATION.

$15 \times 8 \div 12 = 10$ board ft. *Ans.*

SOLUTION. — Since 8 in. is $\frac{8}{12}$ of a foot, we multiply 15 by $\frac{8}{12}$ and obtain the number of board feet, or we may multiply 15 by 8 and divide the result by 12, and the answer is 10 board feet.

2. What are the contents, in board measure, of a joist 16 ft. long, 10 in. wide, and 3 in. thick?

OPERATION.

$\overline{3 \times 10 \times 16} \div 12 = 40$ bd. ft. *Ans.*

SOLUTION. — If the board were 1 in. thick, the square contents would be $16 \times \frac{10}{12}$ or $16 \times 10 \div 12$ board feet; but since it is 3 in. thick it contains 3 times as many board feet or $3 \times 10 \times 16 \div 12 = 40$ board feet.

RULE. — I. *When lumber is not more than* 1 *inch thick, multiply the length in feet by the width in inches, and divide the product by* 12.

II. *When it is more than* 1 *inch thick, multiply the length in feet by the product of the width and thickness in inches, and divide the continued product by* 12.

3. How many board feet are there in 4 boards 16 ft. long, 10 in. wide?

4. How many board feet are there in 2 joists 17 ft. long, 11 in. wide, and 3 in. thick?

5. Find the contents of a board 18 ft. long, 1 ft. 8 in. wide at one end, and 14 in. at the other.

OPERATION.

$\overline{20 \text{ in.} + 14 \text{ in.}} \div 2 = 17$ in.; $\overline{18 \times 17} \div 12 = 25\frac{1}{2}$ bd. ft. *Ans.*

6. Find the cost of 5 boards 12 ft. long, 17 in. wide at one end, and 11 in. at the other, at 6 cents per square foot.

7. How many board feet are there in a stick of timber 36 ft. long, 10 in. thick, 12 in. wide at one end, and 9 in. wide at the other end?

8. Find the cost of 10 planks, each 15 ft. long, 16 in. wide, and $3\frac{1}{2}$ in. thick, at $ 2.25 per hundred feet.

9. Find the cost of 3 pieces of timber, each 26 ft. long and 6 in. by 9 in., at $ 1.75 per hundred board feet.

10. Find the cost of 8 pieces of scantling, 3 in. by 4 in. and 14 ft. long, at $ 9.50 per thousand board feet.

11. Find the cost of 36 boards, 12 ft. long, 11 in. wide, @ $ $2\frac{1}{2}$ per C.

12. Find the cost of 16 planks, $14\frac{1}{2}$ ft. long, 10 in. wide, and 3 in thick, @ \$ $16\frac{1}{4}$ per M.

13. A rectangular field, 16 ch. long and 8 ch. wide, is inclosed by a post and board fence; the posts are set 8 ft. apart, the boards are 16 ft. long, and the fence is 5 boards high. The bottom board is 12 in. wide, the top board is 6 in., and the other three 9 in. wide. The posts cost \$ 25 per C., and the boards \$ 14.80 per M. Required the number of posts, the amount of lumber, and the cost of both.

418. To find the length or width of board feet when one dimension is given.

1. What length of board 8 in. wide contains 10 board feet, and what width of board 15 ft. long contains 10 board feet?

OPERATION.

$\overline{10 \times 12} \div 8 = 15$ ft., length.

$\overline{10 \times 12} \div 15 = 8$ in., width.

SOLUTION. — Since a board foot is the product of the length in feet by the width in inches, divided by 12, if we multiply the product, 10 board feet, by 12 and divide the result by the width in inches, 8, we have the length in feet, 15 ft.; or if we divide the result by the length in feet, 15, we have the width in inches, 8 in.

RULE. — *Multiply the number of board feet by* 12. *Divide the product by the width in inches to obtain the length in feet, or by the length in feet to obtain the width in inches.*

2. What length of board 9 in. wide contains 8 board feet?

3. What length may be cut from a board 15 ft. long and 20 in. wide, so as to leave 15 board feet?

4. What must be the width of a board 16 ft. long, to contain 8 board feet?

5. What must be the width of a piece of board 5 ft. 3 in. long, to contain 7 board feet?

6. What must be the width of a board 18 ft. long to contain 15 board feet?

7. What length may be cut from a board 25 ft. long and 24 in. wide so as to leave 20 board feet?

419. To find the third dimension when the cubic contents and the other two dimensions are given.

1. What length of a piece of timber 6 in. by 9 in., will contain 3 cubic feet?

OPERATION.

$$\overline{1728 \times 3} \div \overline{9 \times 6} = 96;\ 96 \div 12 = 8 \text{ ft., length} \quad \textit{Ans.}$$

RULE. — *Multiply the contents in cubic feet by* 1728; *divide the result by the product of the two known dimensions expressed in square inches; and divide this quotient by* 12 *to express the answer in feet.*

2. A piece of timber is 10 in. by 16 in. What length of it will contain 15 cubic feet?

3. What amount of inch lumber will make a box 4 ft. by 3 ft. 6 in. by 2 ft. 6 in. on the outside?

LOGS.

420. Round Logs are estimated by the square lumber that can be cut from them. The number of board feet in a log 16 ft. or less in length is ascertained as follows:

RULE. — *From the square of the diameter of the smaller end subtract twice this diameter. Take* $\frac{21}{400}$ *of the remainder, and the result will be the number of board feet in a log* 1 *ft. long.*

NOTE. — The diameter of the *smaller* end is always taken as a basis.

Examples.

421. 1. Find the number of board feet in a log 8 ft. long whose smaller diameter is 25 in.

OPERATION.

$$25^2 - (2 \times 25) \times \tfrac{21}{400} \times 8 = 241.5 \text{ board feet} \quad \textit{Ans.}$$

Find the number of board feet in the following logs:

	Length.	Diameter.		Length.	Diameter.
2.	6 ft.	14 in.	6.	9 ft.	19 in.
3.	15 ft.	13 in.	7.	15 ft.	17 in.
4.	16 ft.	24 in.	8.	11 ft.	18 in.
5.	10 ft.	20 in.	9.	12 ft.	16 in.

MASONRY AND PAVING.

422. **Masonry** is estimated by the cubic foot, and by the perch; also by the square foot and the square yard.

423. **Materials** are usually estimated by *cubic* measure; the **work** by *cubic* or *square* measure.

424. A **Perch** of stone or of masonry is $16\frac{1}{2}$ ft. long, $1\frac{1}{2}$ ft. wide, and 1 ft. high, and is equal to 24.75 cu. ft. When stone is built into a wall, 22 cu. ft. make a perch, $2\frac{3}{4}$ cu. ft. being allowed for mortar and filling.

425. **Embankments** and **Excavations** are estimated by the *cubic yard.* A cubic yard of common earth is called a *load.*

426. **Brickwork** is generally estimated by the *thousand bricks;* sometimes in *cubic feet.* In walls, brickwork is estimated at the rate of a brick and a half thick.

North River bricks are 8 in. $\times\ 3\frac{1}{2} \times 2\frac{1}{4}$; Maine bricks are $7\frac{1}{2}$ in. $\times\ 3\frac{3}{8} \times 2\frac{3}{8}$; Philadelphia and Baltimore bricks are $8\frac{1}{4}$ in. $\times\ 4\frac{1}{8} \times 2\frac{3}{8}$; and Milwaukee bricks, $8\frac{1}{2}$ in. $\times\ 4\frac{1}{8} \times 2\frac{3}{8}$.

In estimating *material,* allowance is made for doors, windows, and corners. The length and breadth of a corner are each equal to the thickness of the wall.

In estimating the *work,* masons measure each wall on the *outside,* and ordinarily no allowance is made for doors, windows, and corners; but sometimes an allowance of *one half* is made, this being, however, a matter of *contract.*

Examples.

427. To find the number of bricks in a cubic foot of masonry.

RULE. — I. *Add to the face dimensions of the kind of bricks used, one half the thickness of the mortar or cement in which they are laid, and compute the area.*

II. *Multiply this area by the quotient of the thickness of the wall divided by the number of bricks of which it is composed; the product will be the volume of a brick and its mortar in cubic inches.*

III. *Divide* 1728 *by this volume, and the quotient will be the number of bricks in a cubic foot.*

1. How many Milwaukee bricks are there in a cubic foot of wall $12\frac{3}{4}$ in. thick, laid in courses of mortar $\frac{1}{4}$ of an inch thick?

OPERATION.

$8.5 + (\overline{.25 \times 2} \div 2) = 8.75$ in. = *length* of brick and mortar.
$2.375 + (\overline{.25 \times 2} \div 2) = 2.625$ in. = *height* of brick and mortar.
$8.75 \times 2.625 = 22.96875$ sq. in. = *area* of its face.
$12.75 \div (4.125 + \overline{.25 \times 2} \div 2) = 2.9 +$ bricks, therefore the wall must be 3 bricks thick.
$12.75 \div 3$ (number of bricks in thickness of wall) $= 4.25$ in. = *width* of brick and mortar.
$22.96875 \times 4.25 = 97.617 +$ = *cubic inches* in a brick.
$1728 \div 97.617 + = 17.7 +$ = *number of bricks* in a cubic foot. *Ans.*

2. How many bricks, 8 in. $\times$ 4 $\times$ 2, will be required to build a wall 42 ft. long, 24 ft. high, and $16\frac{1}{2}$ in. thick, laid in courses of mortar $\frac{1}{4}$ of an inch thick?

428. To find the perches of stone required.

RULE.—I. *Multiply the number of cubic feet in the wall, or work to be done, by the number of bricks in a cubic foot; the product will be the number of bricks required.*

II. *Divide the number of cubic feet in the work to be done by* 24.75; *the quotient will be the number of perches.*

1. How many perches of stone, laid dry, will build a wall 60 ft. long, $16\frac{1}{2}$ ft. high, and 18 in. thick?

2. How many perches of masonry are there in a wall 120 ft. long, 6 ft. 9 in. high, and 18 in. thick?

3. What will be the cost of building a wall 60 ft. long, $21\frac{7}{8}$ ft. high, and 17 in. thick, of Philadelphia bricks, laid in courses of mortar $\frac{1}{4}$ of an inch thick, at \$ $12\frac{1}{2}$ per M.?

4. A street 650 ft. long and 72 ft. wide, averages 4.5 ft. below grade. Find the cost of filling it in, at \$.42 a cu. yd.

5. How much will it cost to pave a sidewalk 250 ft. $\times$ 12 ft. with N. R. bricks, 42 to a square yard, laid flat, at \$9 per M.?

6. Find the cost of digging and walling the cellar of a house, whose length is 41 ft. 3 in. and width 33 ft.; the cellar to be 8 ft. deep and the wall $1\frac{1}{2}$ ft. thick, if the excavating costs \$.50 a load, and the stone and mason work \$3.75 a perch.

CAPACITY OF BINS, CISTERNS, ETC.

429. The **Standard Bushel** of the United States contains 2150.42 cubic inches, and is a cylindrical measure $18\frac{1}{2}$ inches in diameter and 8 inches deep (§ 315).

Measures of capacity are all **cubic** measures, solidity and capacity being measured by different units, as seen in the tables.

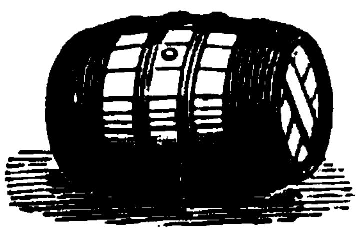
Grain is shipped from New York by the quarter of 480 lb. (8 U.S. bu.), or by the ton of $33\frac{1}{3}$ U.S. bushels.

It is sufficiently accurate in practice to call 5 **stricken measures** equal to 4 **heaped** measures.

Ordinary anthracite coal measures from 36 to 40 cu. ft. to the ton; bituminous coal, from 36 to 45 cu. ft. to the ton.

Lehigh, white ash, egg size, measures about $34\frac{1}{2}$ cu. ft. to the ton (2000 lb.); Schuylkill, white ash, 35 cu. ft.; and gray or red ash, 36 cu. ft. to the ton.

Coal is bought and sold in large quantities by the ton; in small quantities by the bushel, the conventional rate being 28 bu. (5 pecks) to a ton, or about 43.6 cu. ft.



CASK.

430. **Gauging** is the process of finding the capacity or volume of casks and other vessels.

NOTE. — For ordinary purposes the diagonal rod is used, which gives only approximate results.

Examples.

431. To find the exact capacity of a bin in bushels.

RULE. — I. *Divide the contents in cubic inches by* 2150.42; *the quotient will represent the number of bushels.*

Since a standard bushel contains 2150.42 cu. in., and a cubic foot contains 1728 cu. in., a bushel is to a cubic foot nearly as 5 to 4; or a bushel is equal to $1\frac{1}{4}$ cu. ft. nearly. Hence for all practical purposes,

II. *Any number of cubic feet diminished by* $\frac{1}{5}$ *will represent an equivalent number of bushels.*

Thus, 250 cu. ft. $-\ \frac{1}{5}$ of 250 cu. ft. $= 200$, the number of bushels in 250 cu. ft.

III. *Any number of bushels increased by* $\frac{1}{4}$ *will represent an equivalent number of cubic feet.*

Thus, 200 bu. + $\frac{1}{4}$ of 200 bu. = 250, the number of cubic feet in 200 bushels.

1. How many bushels of wheat can be put into a bin 8 ft. long, 6 ft. 6 in. wide, and 3 ft. 4 in. deep?

2. What must be the depth of a bin to contain 240 bu., its length being 10 feet and its width 5 feet?

OPERATION.

240 bu. + 60 bu. = 300. $300 \div \overline{10 \times 5} = 6$ ft., the depth *Ans.*

RULE. — *Divide the contents in cubic feet or inches by the product of two dimensions in the same denomination.*

3. What must be the length of a bin that is 6 feet wide and $4\frac{1}{2}$ feet deep to contain 324 bushels?

4. How many bushels of apples will a bin hold that is 12 ft. long, 3 ft. wide, and 2 ft. 6 in. deep? How many of barley?

5. A bin 20 ft. long, 12 ft. wide, and 5 ft. deep is full of wheat. What is its value at \$2 a bushel?

6. A bin 7 ft. long, 6 ft. wide, and 5 ft. deep is $\frac{3}{4}$ full of rye. What is its value at \$1.37$\frac{1}{2}$ a bushel?

7. A crib, the inside dimensions of which are 15 ft. by 7 ft. 4 in. by 8 ft., is full of corn in the ear. If 2 bushels of ears make 1 bushel of shelled corn, what is the value of the whole, when shelled, at \$.92 a bushel?

8. If 1 bu. or 60 lb. of wheat make 48 lb. of flour, how many barrels of flour can be made from the contents of a bin 10 ft. long, 5 ft. wide, and 4 ft. deep, filled with wheat?

9. How many tons of red ash coal, egg size, will a bin 17 ft. long, 6 ft. wide, and 3 ft. deep contain?

10. Dunkley & Co. bought 12400 bu. of wheat, delivered in New York at \$1.50 a bushel. They shipped the same to Liverpool, paying 6*s.* sterling per quarter freight, and sold the entire cargo at 12*s.* per cental. Making no allowance for exchange or for waste, what was the gross gain in U.S. money, the £ being valued at \$4.866$\frac{1}{2}$?

11. A bin 6 ft. long, 4 ft. deep, and 5 ft. 9 in. wide is full of Lehigh white ash coal. Find its value at $ 6.75 a ton.

12. Find its value if full of Schuylkill white ash, at $5.90 a ton. If full of Schuylkill red ash at $ 5.50 a ton.

13. What is the weight of 225 cu. ft. of water?

432. To find the exact capacity of a vessel or space in gallons.

RULE.—*Divide the contents in cubic inches by* 231 *for liquid gallons, or by* 268.8 *for dry gallons.*

1. How many gallons of water will a cistern hold, that is 4 feet square and 6 feet deep?

OPERATION.

$(4 \times 4 \times 6 \times 1728) \div 231 = 718\frac{10}{77}$ gal., capacity *Ans.*

2. How many cubic feet are there in a space that holds 48 hhd.?

3. How many hogsheads will a cistern 11 ft. long, 6 ft. wide, and 7 ft. deep contain?

4. How many gallons will a space contain that is 22.5 ft. long, 3.25 ft. wide, and 6.4 ft. deep?

5. A man constructed a cistern to hold 32 hogsheads, the bottom being 6 ft. by 8 ft. What was its depth?

6. A tank in the attic of a house is 6 ft. 6 in. long, 4 ft. wide, and 3 ft. 6 in. deep. How many gallons of water will it hold?

7. If 64 quarts of water are put into a vessel that will exactly hold 64 quarts of wheat, how much will the vessel lack of being full?

8. If a man buys 10 bu. of chestnuts at $ 5 a bushel, dry measure, and sells the same at 25 cents a quart, liquid measure, how much does he gain?

9. A cistern 5 ft. by 4 ft. by 3 ft. is full of water. If it is emptied by a pipe in 1 hr. 30 min., how many gallons are discharged through the pipe in a minute?

10. A vat that will hold 5000 gallons of water, will hold how many bushels of corn?

11. A cellar 40 ft. long, 20 ft. wide, and 8 ft. deep is half full of water. What will be the cost of pumping it out, at 6 cents a hogshead?

12. A reservoir 24 ft. 8 in. long by 12 ft. 9 in. wide is full of water. How many cubic feet must be drawn off to sink the surface 1 foot? How many gallons?

433. To find the volume of a cask in gallons.

RULE. — I. *To find approximately the mean diameter of a cask, add to the head diameter* $\frac{2}{3}$, *or, if the staves are but little curved*, .6 *of the difference between the head diameter and the bung diameter.*

II. *To find the volume of a cask in gallons, multiply the square of the mean diameter by the length* (*both in inches*), *and this product by* .0034.

1. How many gallons are there in a cask whose head diameter is 24 inches, bung diameter 30 inches, and its length 34 inches?

OPERATION.

$$24 + (\overline{30 - 24} \times \tfrac{2}{3}) = 28 \text{ in., mean diameter.}$$
$$28^2 \times 34 \times .0034 = 90.63 \text{ gal., capacity } \textit{Ans.}$$

2. What is the volume of a cask whose length is 40 in., the diameters being 21 and 30 in., respectively?

3. How many gallons are there in a cask of slight curvature, 3 ft. 6 in. long, the head diameter being 26 in., and the bung diameter 31 in.?

4. Find the number of gallons in a cask with curved staves, the head diameter being 4 ft., bung diameter 4 ft. 6 in., and length 6 ft. 6 in.

5. What are the number of gallons in a cask having straight staves from bung to head, the length being 20 in., the bung diameter 15 in., and head diameter 12 in.?

PLASTERING, PAINTING, KALSOMINING.

434. Plastering, painting, and kalsomining are generally computed by the square yard.

The processes of calculating the cost of plastering, painting, and kalsomining vary so much in different localities that it is impossible to lay down any rule. Usually some allowance is made for doors, windows, etc., on which no work is done; but sometimes the measurements of walls are taken regardless of such openings. At other times, one half the area of the openings is deducted.

Examples.

435. 1. How much would it cost to plaster the walls and ceiling of a room 16 ft. long, 15 ft. wide, and 11 ft. high, in which there are 2 doors, 2 ft. 8 in. by 7 ft., and 2 windows, 2 ft. 10 in. by 6 ft. 2 in., at \$ .37 a sq. yd., deducting half the area for doors and windows?

OPERATION.

$2 \times 16 \times 11 = 352$
$2 \times 15 \times 11 = 330$
$16 \times 15 = \underline{240}$
Total area, 922 sq. ft.
$2 \times 2\frac{2}{3} \times 7 = 37.3$
$2 \times 2\frac{5}{6} \times 6\frac{1}{6} = \underline{34.9}$
$72.2 \div 2 = 36.1$
sq. ft. area to be deducted.
$922 - 36.1 = 885.9$ sq. ft.
$885.9 \div 9 = 98.4$ sq. yd.
$98.4 \times \$.37 = \$36.41$ *Ans.*

SOLUTION. — Two of the walls will be 16 × 11 ft., two, 15 × 11 ft., and the ceiling, 16 × 15 ft. Adding these, we find the total area to be 922 sq. ft. The area of the two doors will be twice $7 \times 2\frac{2}{3}$ ft., and of the two windows twice $2\frac{5}{6} \times 6\frac{1}{6}$ ft. Adding these, the area of doors and windows is 72.2. We deduct half this area, 36.1, from 922, divide the remainder, 885.9 sq. ft., by 9, to reduce it to square yards, multiply \$ .37 by the result, 98.4 sq. yd., and find the answer \$ 36.41.

2. How much would it cost to plaster the walls and ceiling of a room 12 ft. long, 17 ft. wide, and 10 ft. high, at \$ .35 a sq. yd., making no allowance for openings?

3. How much will it cost to kalsomine a hall 6 ft. 6 in. by 27 ft. by 11 ft., with two coats, at 3¢ a sq. yd. each?

4. How much will it cost to paint the walls and ceiling of a room 18 ft. long, 14 ft. wide, 9 ft. high, with 3 coats, at \$ .07 a sq. yd. each, making no allowance for openings?

PAPERING.

436. Wall paper is sold only by the roll, and any part of a roll is considered a whole roll.

American paper is commonly $\frac{1}{2}$ a yard wide, and has 8 yards in a roll. Foreign papers vary in width and length to the roll. Borders and friezes are sold by the yard, and vary in width from 3 inches to 18 inches.

Paper is also often put up in double rolls which are 16 yards long and therefore equal to two single rolls.

It is not possible to find in advance the exact cost of papering a room, since there is frequently much waste, and a paper hanger will charge for the number of rolls actually used in doing the work; but it is well to make an approximate estimate.

Examples.

437. 1. How much would it cost to paper the walls of a room 14 feet long, 12 feet wide, and 8 feet high from baseboard to ceiling, with paper 8 yards long, and $\frac{1}{2}$ a yard wide, costing \$.30 a roll?

OPERATION.

$2 \times 14 = 28$
$2 \times 12 = \underline{24}$
$\qquad\quad 52$ ft.
$52 \div 3 = 14$ yd. $=$
28 half-yards, or strips.
$24 \div 8 = 3$ strips to a roll.
$28 \div 3 = 9+$, or 10 rolls.
\$.30 $\times 10 =$ \$3.00 *Ans.*

SOLUTION. — The distance around the room is 2×14 ft. $+ 2 \times 12$ ft. $= 52$ ft. $= 14$ yd. $= 28$ half-yards, the number of strips required. Since the height is 8 ft., and there are 24 ft. on a roll, 1 roll will make 3 strips; and it will take as many rolls to make 28 strips as 3 is contained times in 28, which is 9+. Therefore, 10 rolls will be required. At \$.30 a roll, they will cost \$3.00.

RULE. — I. *Find the entire distance around the room in yards. Multiply this by 2 to find the number of half-yards, or strips, since the paper is only half a yard wide.*

II. *Divide the number of half-yards by the number of strips that can be cut from a roll, and the result will be the number of rolls required.*

NOTE. — Since there are 24 ft. in a roll 8 yd. long, if the distance from baseboard to ceiling is 8 ft. or less, 3 strips can be cut from a roll; if more than 8 ft., and not more than 12 ft., 2; etc. In the former case the divisor would be 3; in the latter, 2.

2. How many double rolls of paper 16 yd. long, ½ yd. wide, will be required to paper the walls of a room 30 ft. long, 30 ft. wide, and 12 ft. from baseboard to ceiling?

3. Find the cost of papering a room 16 ft. long, 15 ft. wide, and 11 ft. high, with American paper 8 yd. long, and ½ yd. wide, at $.50 a roll, allowing 1 ft. for baseboard, and making no deductions for openings.

4. How many rolls of paper 8 yd. long and ½ yd. wide will be required to paper the walls of a room 25 ft. long and 15 ft. wide, and 10 ft. from baseboard to ceiling, allowing for two doors, 9 ft. by 4 ft., and one window, 6 ft. by 7 ft.?

5. At $.35 a roll, what will be the cost of papering the walls and ceiling of a room 18 ft. long, 14 ft. wide, and 10 ft. high, making no allowance for openings, the baseboard being 9 in. high?

6. How many rolls of paper, 8 yd. long and ½ yd. wide, will be required to paper the walls of a room 20 ft. long, 16 ft. wide, and having a height of 10 ft. from the baseboard to the ceiling, allowing for one door 3 ft. by 8 ft., and for two windows, each 3 ft. by 6½ ft.?

7. Find the cost of papering a room 20 ft. long, 18 ft. wide, 10 ft. high from baseboard, with paper 20 in. wide, 16 yd. in a roll, at $3.25 a roll, and a border at 25 cents per running foot; allowing for a fireplace 4½ ft. by 3½ ft., a door 7 ft. by 4 ft., and 2 windows, each 6 ft. by 4 ft., but making no allowance for border.

8. Find the cost of papering a room 25 ft. long, 19 ft. wide, 10 ft. high from baseboard to ceiling, with paper ½ yd. wide, 8 yd. in a roll, at $.85 a roll; allowing for 2 doors, each 7 ft. high, 3 ft. wide, and for 3 windows, each 5 ft. 9 in. high and 3½ ft. wide.

9. Find the cost of papering a room 18½ ft. long, 16½ ft. wide, and 12 ft. high with American paper 8 yd. long and ½ yd. wide, at $.75 a roll, allowing 1 ft. for baseboard, and making a deduction of $3.00 for doors and openings.

CARPETING.

438. Carpets are usually 1 yd. wide or $\frac{3}{4}$ yd. wide, and are sold by the yard.

Note. — We cannot often estimate the amount needed by finding the square yards in a floor, as there may be waste in matching or in turning under.

Examples.

439. 1. How many yards of carpet $\frac{3}{4}$ yd. wide will be needed for a room 20 ft. long by $13\frac{1}{2}$ ft. wide, if the strips run lengthwise? How much if the strips run across the room?

OPERATION.

$13\frac{1}{2}$ ft. $= 162$ in. 20 ft. $= 240$ in.
$162 \div 27 = 6$ strips.
$240 \div 36 = 6\frac{2}{3}$, length of strips.
$6 \times 6\frac{2}{3} = 40$ yd., lengthwise. *Ans.*
$240 \div 27 = 8 + = 9$ strips.
$162 \div 36 = 4\frac{1}{2}$, length of strips.
$9 \times 4\frac{1}{2} = 40\frac{1}{2}$ yd., crosswise. *Ans.*

Solution. — Since the carpet is 27 in. wide and the room $13\frac{1}{2}$ ft. or 162 in. wide, when laid lengthwise we shall need 6 breadths of carpet of the required length. And since the room is 20 ft. or 240 in. long, it will take as many yards for each breadth as 36 in., or 1 yd., is contained times in 240 in., or $6\frac{2}{3}$ yd. Hence for 6 breadths, or strips, it will require $6 \times 6\frac{2}{3}$ yd., or 40 yd.

If laid crosswise, we shall need as many strips as 27 in., the width of the carpet, is contained times in 240 in., the length of the room, which is $8\frac{8}{9}$; therefore we shall need 9 strips, and $\frac{1}{9}$ will be turned under. We shall need as many yards for each strip as 36 in., or 1 yd., is contained in 162 in., which is $4\frac{1}{2}$ yd. Hence for 9 strips we need 9 times $4\frac{1}{2}$ yd., or $40\frac{1}{2}$ yd.

Rule. — *Find the number of breadths or strips required, and the length of each strip.*

2. How many yards of carpet, 1 yd. wide, will be needed for a room 18 ft. square, if the strips run lengthwise? Crosswise?

3. Find the cost of carpeting a room $16\frac{1}{2}$ ft. long and 15 ft. wide with carpet $\frac{3}{4}$ yd. wide, lengthwise, at $1.30 per yard.

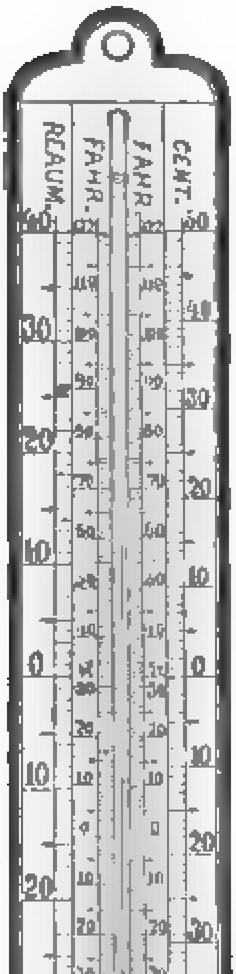
4. Find the cost of a carpet $\frac{3}{4}$ yd. wide, at $1.50 per yard, for a room 17 ft. long and $14\frac{1}{2}$ ft. wide, there being a waste of 1 yd. in matching the pattern, and the carpet layer having been instructed to lay the carpet in the most economical way.

5. How many meters of carpet 8^{dm} wide will be required for a floor 7^{m} long and 3^{m} wide?

TEMPERATURE.

440. Since most bodies expand in proportion to the amount of heat applied, we measure heat or temperature by means of a **Thermometer**, which registers in degrees the expansion of a bulb of mercury, or alcohol, etc.

The three thermometers in general use are the **Fahrenheit** (F.), the **Centigrade** (C.), and the **Réaumur** (R.), each of which is based on a different scale.



Combination Thermometer showing the Fahrenheit, Centigrade, and Réaumur Scales.

The **Fahrenheit** thermometer is so graduated that the freezing point of water is at 32° above the zero of the scale and the boiling point at 212° above zero, the distance between being 180°. It is commonly used in the United States and in England.

In the **Centigrade** thermometer the freezing point of water is at zero and the boiling point at 100° above zero.

In the **Réaumur** thermometer the freezing point is at zero and the boiling point at 80° above zero. Hence,

$1°$ F. $= \frac{100}{180}°$ C. $= \frac{5}{9}°$ C. and $\frac{80}{180}°$ R. $= \frac{4}{9}°$ R.

$1°$ C. $= \frac{180}{100}°$ F. $= \frac{9}{5}°$ F. and $\frac{80}{100}°$ R. $= \frac{4}{5}°$ R.

$1°$ R. $= \frac{180}{80}°$ F. $= \frac{9}{4}°$ F. and $\frac{100}{80}°$ C. $= \frac{5}{4}°$ C.

Examples.

441. To reduce from one scale to another.

1. Express 98° Fahrenheit in the Centigrade scale.

OPERATION.

$98° - 32° = 66°$

$66 \times \frac{5}{9}°$ C. $= 36\frac{2}{3}°$ C. *Ans.*

SOLUTION. — 98° F. is 66° above the freezing point. Since 1° F. = $\frac{5}{9}$°C., 66° F. = 66 × $\frac{5}{9}$° C. = $36\frac{2}{3}$° C. *Ans.*

2. Express 25° Centigrade in the Fahrenheit scale.

OPERATION.

$25 \times \frac{9}{5}°$ F $= 45°$ F. above freezing point.

$45° + 32° = 77°$ F. *Ans.*

SOLUTION. — Since 1° C. = $\frac{9}{5}$° F., 25° C. = 25 × $\frac{9}{5}$° F. = 45° F. above freezing point. 32° + 45° = 77° F. *Ans.*

3. Express 98° F. in the R. scale, and 24° R. in the F. scale.

OPERATION.

$98° - 32° = 66°$ F. above freezing point.
$66 \times \frac{4}{9}°$ R. $= 29\frac{1}{3}°$ R. *Ans.*
$24 \times \frac{9}{4}°$ F. $= 54°$ F. above freezing point.
$54° + 32° = 86°$ F. *Ans.*

SOLUTION. — 98° F. is 66° above freezing point. 1° F. = $\frac{4}{9}°$ R. $66 \times \frac{4}{9}°$ R. $= 29\frac{1}{3}°$ R. *Ans.* 1° R. = $\frac{9}{4}°$ F. $24 \times \frac{9}{4}°$ F. = 54° F. above freezing point. $54° + 32° =$ 86° F. *Ans.*

4. Express 15° C. in the R. scale, and 12° R. in the C. scale.

OPERATION.

$15 \times \frac{4}{5}°$ R $= 12°$ R. *Ans.*
$12 \times \frac{5}{4}°$ C. $= 15°$ C. *Ans.*

SOLUTION.—The freezing point is the same in these two scales. 1° C. = $\frac{4}{5}°$ R.; $15 \times \frac{4}{5}°$ R. = 12° R. *Ans.* 1° R. = $\frac{5}{4}°$ C.; $12 \times \frac{5}{4}°$ C. = 15° C. *Ans.*

RULE. — I. *To change from F. to C., subtract* 32° *and multiply by* $\frac{5}{9}$; *from F. to R., subtract* 32° *and multiply by* $\frac{4}{9}$.

II. *To change from C. to F., multiply by* $\frac{9}{5}$ *and add* 32°; *from C. to R., multiply by* $\frac{4}{5}$.

III. *To change from R. to F., multiply by* $\frac{9}{4}$ *and add* 32°; *from R. to C., multiply by* $\frac{5}{4}$.

NOTE. — The minus sign is used in the examples to indicate degrees below zero.

Express in the Centigrade scale:

5. 212° F.	7. − 20° F.	9. 80° F.	11. −3° F.
6. 80° R.	8. − 20° R.	10. 60° F.	12. −35° R.

Express in the Fahrenheit scale:

13. 100° C.	15. 0° C.	17. −10° C.	19. 95° R.
14. 80° R.	16. 0° R.	18. −15° R.	20. 72° C.

Express in the Réaumur scale:

21. 60° F.	23. − 25° F.	25. −30° C.	27. 212° F.
22. 65° C.	24. 80° C.	26. 100° F.	28. 100° C.

29. On a day in July a Centigrade thermometer registered 35° in the sun. Express this by the Fahrenheit scale; by the Réaumur scale.

30. The record of a patient's temperature by the Fahrenheit thermometer was as follows: 9 P.M., $98\frac{1}{2}°$; 10 P.M., 99°; 11 P.M., 100°; 12 M., $98\frac{3}{4}°$. Find his average temperature for the four hours in each of the three scales.

SPECIFIC GRAVITY.

442. The specific gravity of a substance is its weight compared with an equal bulk of water. Thus, if a substance is twice as heavy as water, its specific gravity is 2, etc.

The following table gives the specific gravity of a number of substances:

Liquids.		Metals and Stones.		Sundries.	
Water	1.00	Granite	2.78	Indigo	.77
Alcohol, pure	.79	Diamond	3.53	Ice	.92
Olive Oil	.92	Zinc	6.91	Gunpowder	.93
Turpentine	.99	Cast Iron	7.21	Butter	.94
Wine	1.00	Bar Iron	7.79	Clay	1.20
Cider	1.02	Tin	7.29	Coal	1.30
Cow's Milk	1.03	Steel	7.83	Opium	1.34
Timber.		Brass	8.40	Honey	1.45
Cork	.24	Copper	8.95	Ivory	1.83
Poplar	.38	Silver	10.47	Sulphur	2.03
Maple	.75	Lead	11.35	Porcelain	2.26
Beech	.85	Mercury	13.57	Marble	2.70
Mahogany	1.06	Gold	19.26	Chalk	2.79
Oak	1.17	Platinum	22.07	Glass	2.89

443. A cubic foot of water weighs $62\frac{1}{2}$ lb. or 1000 oz. avoirdupois. A cubic centimeter of water weighs 1 gram. A cubic decimeter or liter of water weighs 1 kilogram. A cubic meter of water weighs a tonneau or metric ton.

Principle. — *The specific gravity of water or of any substance is the same as the number of grams in a cubic centimeter of the substance, the number of kilograms in a cubic decimeter, or liter, or the number of metric tons in a cubic meter.*

444. To find the specific gravity of a substance we must know the weight of the same volume of water. When a substance heavier than water is immersed in water, the water buoys it up or makes it as much lighter as the weight of the

volume of water it displaces. Hence it is easy to find the weight and volume of water displaced.

Examples.

445. To find the volume of a substance when its weight in air and water are given.

1. A piece of lead weighs $1418\frac{3}{4}$ lb. when suspended in air. In water it weighs $1293\frac{3}{4}$ lb. What is its volume?

OPERATION.

$1418\frac{3}{4}$ lb. $-$ $1293\frac{3}{4}$ lb. $= 125$ lb.

$125 \div 62\frac{1}{2} = 2$ cu. ft. *Ans.*

SOLUTION. — The weight of water displaced $= 1418\frac{3}{4}$ lb. $-$ $1293\frac{3}{4}$ lb. $=$ 125 lb., and since $62\frac{1}{2}$ lb. $= 1$ cu. ft., 125 lb. $= 125 \div 62\frac{1}{2} = 2$ cu. ft. Since the lead displaces 2 cu. ft. of water, its own volume must be 2 cu. ft.

2. A piece of lead suspended in air weighs 283.75^g. In water it weighs 258.75^g. What is its volume?

OPERATION.

$283.75^g - 258.75^g = 25^g$

$25^g = 25^{cu\,cm}$ *Ans.*

SOLUTION. — The weight of water displaced is 25^g, and since 1^g of water has a volume of $1^{cu\,cm}$, $25^g = 25^{cu\,cm}$.

RULE. — I. *Find the difference in pounds between the weight in air and water, and divide this by $62\frac{1}{2}$. The result will be the volume of the substance in cubic feet.*

II. *Find the difference between the weight in air and water expressed in grams, kilograms, or tonneaus. The result will be the same as the volume expressed in cubic centimeters, cubic decimeters, or cubic meters, respectively.*

NOTE. — If the difference between the weight in air and water is expressed in ounces, divide this difference by 1000, and the quotient will be the volume in cubic feet.

3. A quantity of rock salt weighs 4.52^T in air, and 2.52^T in water. What is its volume?

4. A quantity of brass weighs 8400 oz. in air, and 7400 oz. in water. What is its volume?

5. What is the volume of a piece of brass which weighs 4200 lb. in air, and 3700 lb. in water?

6. If a piece of quartz suspended in air weighs 92.75^g, and in water 57.75^g, what is its volume?

446. To find the specific gravity of a substance when its weight in air and water are given.

1. What is the specific gravity of the lead in Ex. 1, § 445?

OPERATION.

$$1418\tfrac{3}{4}\text{ lb.} - 1293\tfrac{3}{4}\text{ lb.} = 125\text{ lb. water displaced.}$$

$$1418\tfrac{3}{4} \div 125 = 11.35\text{ Sp. G. } \textit{Ans.}$$

SOLUTION. — The water displaced = 125 lb., and since a volume weighing $1418\frac{3}{4}$ lb. displaces an equal volume of water weighing 125 lb., the substance must be $1418.75 \div 125 = 11.35$ times as heavy as water. Hence its specific gravity must be 11.35.

2. What is the specific gravity of the lead in Ex. 2, § 445?

OPERATION.

$$283.75^g - 258.75^g = 25^g$$

$$283.75 \div 25 = 11.35\text{ Sp. G. } \textit{Ans.}$$

SOLUTION. — The water displaced = 25 grams, and since a volume weighing 283.75^g displaces 25^g of water, it must be $283.75 \div 25 = 11.35$ times as heavy as water.

RULE. — *Divide the weight in air by the loss of weight in water.*

3. A piece of brass which weighs 4200 lb. in air weighs 3700 lb. in water. What is its specific gravity?

4. A piece of quartz suspended in the air weighs 92.75^g; in the water it weighs 57.75^g. What is its specific gravity?

5. What is the specific gravity of a quantity of rock salt weighing 4.52^T in air, and 2.52^T in water?

6. What is the specific gravity of a quantity of brass weighing 8400 oz. in air, and 7400 oz. in water.

447. To find the specific gravity of a substance when its weight and volume are given.

1. A block of marble containing 5 cu. ft. weighs 843.75 lb. What is the specific gravity of marble?

OPERATION.

$$5 \times 62.5\text{ lb.} = 312.5\text{ lb.}$$

$$843.75 \div 312.5 = 2.7\text{ Sp. G. } \textit{Ans.}$$

SOLUTION. — 5 cu. ft. of water weigh 5×62.5 lb. = 312.5 lb. Hence marble is as many times as heavy as water as 312.5 is contained times in 843.75, which is 2.7, the specific gravity.

2. What is the specific gravity of marble if $25^{\text{cu cm}}$ weigh 67.5^{g}?

OPERATION.

$67.5 \div 25 = 2.7$ Sp. G. *Ans.*

SOLUTION. — $25^{\text{cu cm}}$ of water weigh 25^{g} (§ 443). Hence marble must be as many times as heavy as water as 25 is contained times in 67.5, which is 2.7, the specific gravity.

RULE. — I. *If the weight and volume are expressed in pounds and cubic feet, multiply $62\frac{1}{2}$ lb. by the number expressing the volume, and divide the given weight by the product.*

II. *If the weight and volume are expressed in grams and cubic centimeters, divide the number expressing the weight by that expressing the volume.*

NOTES. — 1. If the weight is expressed in ounces, multiply the volume in cubic feet by 1000, and divide the given weight by the product.

2. If the weight is expressed in kilograms, divide by the number expressing the volume in cubic decimeters; if in tonneaus, by the number expressing the volume in cubic meters (§ 448, Prin.).

Find the specific gravity of:

3. Ether, if $2\frac{1}{2}$ cu. ft. weigh $112\frac{1}{2}$ lb.

4. Naphtha, if $5^{\text{cu cm}}$ weigh 4.25^{g}.

5. Marble, if $56.64^{\text{cu dm}}$ weigh 152.928^{Kg}.

6. Ebony, if 8 cu ft. weigh 595 lb.

7. If a decaliter of sea water weighs 10.3^{Kg}, what is its specific gravity?

8. If 13 cu. ft. of chalk weigh $2266\frac{1}{8}$ lb., what is the specific gravity of chalk?

9. If 8 cu. ft. of lead weigh 5675 lb., what is the specific gravity of lead?

448. To find the weight of a substance when its volume and specific gravity are given.

1. Find the weight of 16 cu. ft. of copper, the specific gravity being 8.95.

OPERATION.

16×62.5 lb. $\times 8.95 = 8950$ lb. *Ans.*

SOLUTION. — Since 1 cu. ft. of water weighs $62\frac{1}{2}$ lb. 16 cu. ft. will weigh $16 \times 62\frac{1}{2}$ lb.; and since copper is 8.95 as heavy as water, 16 cu. ft. of copper will weigh $16 \times 8.95 \times 62.5$ lb. $=$ 8950 lb.

2. Find the weight of $120^{cu\ cm}$ of copper.

OPERATION.

$120^{g} \times 8.95 = 1074^{g}$ *Ans.*

SOLUTION. — Since $1^{cu\ cm}$ of water weighs 1^{g}, $120^{cu\ cm}$ weigh 120^{g}, and since copper is 8.95 times as heavy as water, the same amount of copper will weigh $8.95 \times 120^{g} = 1074^{g}$.

RULE. — I. *If the volume is expressed in cubic feet, multiply $62\frac{1}{2}$ lb. by the number expressing the volume, and the product by the specific gravity. The result will be the weight in pounds.*

II. *If the volume is expressed in cubic centimeters, multiply the number expressing the volume by the specific gravity. The result will be the weight in grams.*

NOTES. — 1. If the volume in cubic feet is multiplied by 1000 and the product by the specific gravity, the result will be the weight in ounces.

2. If the volume is expressed in cubic decimeters, the result after multiplying by the specific gravity will be the weight in kilograms; if in cubic meters, the result will be the weight in tonneaus (§ 443, Prin.).

3. Consult the table on p. 264 for specific gravities not given.

3. Find the weight in pounds of $22\frac{1}{2}$ cu. ft. of turpentine, its specific gravity being .99.

4. The specific gravity of maple wood is .75. What is the weight in pounds of 780 cu. ft.

5. The specific gravity of bar iron is 7.79. Find the weight in tonneaus of $925^{cu\ m}$ of it.

6. Find the weight in ounces of $25^{cu\ cm}$ of opium, its specific gravity being 1.34.

7. How many ounces will 25 cu. in. of sulphur weigh?

8. What will be the weight in kilograms of $650^{cu\ dm}$ of butter?

9. Find the weight in kilograms of a bar of silver 4^{dm} long, 1.5^{dm} wide, and 1^{dm} thick.

10. What is the weight of 15 cu. ft. of alcohol?

11. Which is the heavier and how much, 10 cu. ft. of olive oil or 8 cu. ft. of cider?

12. Find the weight in grams of $5000^{cu\ cm}$ of cork.

13. Find the weight in pounds of 19 cu. ft. of lead.

449. To find the volume of any substance when its weight and specific gravity are given.

1. Find the number of cubic feet in a block of marble weighing 843.75 lb., the specific gravity of marble being 2.7.

OPERATION.

2.7×62.5 lb. $= 168.75$ lb.

$843.75 \div 168.75 = 5$ cu. ft. *Ans.*

SOLUTION. — Since one cubic foot of marble weighs 2.7×62.5 lb., or 168.75 lb., 843.75 lb. will be the weight of as many cu. ft. as 168.75 is contained times in 843.75, or 5 cu. ft.

2. How many cubic centimeters are there in a piece of marble weighing 675^{g}?

OPERATION.

$675 \div 2.7 = 250^{cu\,cm}$ *Ans.*

SOLUTION. — Since marble is 2.7 times as heavy as water, $1^{cu\,cm}$ of marble weighs 2.7^{g} (§ 443, Prin.), and 675^{g} must be the weight of as many cubic centimeters as 2.7 is contained times in 675, or $250^{cu\,cm}$.

RULE. — I. *If the weight is expressed in pounds, multiply $62\frac{1}{2}$ lb. by the specific gravity and divide the number expressing the weight by the product. The quotient will be the volume expressed in cubic feet.*

II. *If the weight is expressed in grams, divide the number expressing the weight by the specific gravity; the quotient will be the volume expressed in cubic centimeters.*

NOTES. — 1. If the weight is expressed in ounces, multiply the specific gravity by 1000, divide the weight by the result, and the quotient will be the answer in cubic feet.

2. If the weight is expressed in kilograms, divide by the specific gravity and the quotient will be in cubic decimeters. If the weight is expressed in tonneaus, divide by the specific gravity and the quotient will be the answer in cubic meters.

Find the volume of:

3. 350 lb. of steel.

4. 875^{Kg} of indigo.

5. 3^{T} of gunpowder.

6. 32^{T} of cast iron.

7. 6112 oz. of copper.

8. 465^{Kg} of chalk.

9. How many cubic centimeters are there in a piece of gold weighing 77.04^{g}?

10. How many decaliters of alcohol will weigh 869^{Kg}?

11. The specific gravity of ivory being 1.83, find the volume of a quantity of ivory weighing $800\frac{5}{8}$ lb.

RATIO.

450. Ratio is the measure of the relation between two like numbers with respect to comparative value.

NOTE. — There are two methods of comparing numbers with respect to value; 1st, by subtracting one from the other; 2d, by dividing one by the other. The relation expressed by the difference is sometimes called *arithmetical ratio*, and the relation expressed by the quotient, *geometrical ratio*.

When one number is compared with another, as 12 with 4, by means of division, thus, $12 \div 4 = 3$, the quotient, 3, shows the relative value of the dividend when the divisor is considered as a *unit* or *standard*. The ratio in this case shows that 12 is 3 times 4; that is, if 4 is regarded as a unit, 12 will be 3 units, or the relation of 12 to 4 is that of 3 to 1.

451. Ratio is indicated in two ways:

1. By placing two points between the two numbers compared, writing the divisor before and the dividend after the points. Thus, the ratio of 8 to 24 is written 8 : 24; the ratio of 7 to 5 is written 7 : 5.

2. In the form of a fraction. Thus, the ratio of 8 to 24 may be written $\frac{8}{24}$; of 7 to 5, $\frac{7}{5}$.

452. The **Terms** of a ratio are the numbers compared. The first term or the dividend is the **Antecedent**; the second term or the divisor is the **Consequent**. The two terms together form a **Couplet**.

NOTE. — The antecedent is the number to be compared with the consequent as a unit or standard.

453. The **Value** of a ratio is the *quotient* of the antecedent divided by the consequent, and is an abstract number.

Thus, in the ratio \$24 : \$6, \$24 is the antecedent, \$6 is the consequent, and 4, the quotient of \$24 ÷ \$6, is the *value* of the ratio.

NOTE. — Some authorities claim that the value of the ratio is found by dividing the consequent by the antecedent. This is erroneously called the *French method*, as the most eminent French mathematicians teach the reverse. The weight of argument is in favor of the old method given above.

454. A **Direct Ratio** arises from dividing the antecedent by the consequent.

455. An **Inverse** or **Reciprocal Ratio** is obtained by dividing the consequent by the antecedent. The reciprocal of a ratio equals 1 divided by the ratio.

Thus, the *direct* ratio of 5 to 15 is $\frac{5}{15} = \frac{1}{3}$; and the *inverse* ratio of 5 to 15 is $\frac{15}{5} = 3$.

NOTES. — 1. When the numerator and denominator of a fraction are interchanged, the fraction is said to be inverted; and in the same way, when the antecedent and consequent are interchanged, the ratio is called an *inverse ratio*.

2. One quantity is said to *vary directly* as another, when the two increase or decrease together in the same ratio; and one quantity is said to *vary inversely* as another, when one increases in the same ratio as the other decreases. Thus time varies *directly* as wages; that is, the greater the time the greater the wages, and the less the time the less the wages. Again, velocity varies *inversely* as the time, the distance being fixed; that is, in traveling a given distance, the greater the velocity the less the time, and the less the velocity the greater the time.

456. A **Simple Ratio** is the ratio between two terms and consists of a single couplet; as $3:12$.

457. A **Compound Ratio** is the ratio of the corresponding terms of two or more simple ratios. It consists of two or more couplets the products of whose corresponding terms form a simple ratio.

When the multiplication is performed in a compound ratio, the result is a simple ratio. Thus, the compound ratio formed from the simple ratios, $8:4$ and $9:12$ may be expressed

$$(8:4) \times (9:12) \text{ or, } 8 \times 9 : 4 \times 12 = 72:48, \text{ or}$$

$$\left.\begin{matrix} 8:\ 4 \\ 9:12 \end{matrix}\right\} = 72:48, \text{ or } \frac{8}{4} \times \frac{9}{12} = \frac{3}{2} = 3:2.$$

458. Ratio can exist only between like numbers or between two quantities of the same kind. In the comparison of like numbers we observe that:

If the numbers are *simple*, whether abstract or concrete, their ratio may be found directly by division.

Compound denominate numbers must first be reduced to the same unit or denomination.

If the numbers are *fractional*, and have a common denominator, the fractions will be to each other as their numerators; if they have not a common denominator, their ratio may be found either directly by dividing the first by the second, or by reducing them to a common denominator and comparing their numerators.

NOTE.—Though ratio can exist only between numbers of the same kind, one of two unlike numbers may *vary directly* or *inversely* as the other. Thus cost varies directly as quantity in the purchase of goods, time varies inversely as velocity in the descent of falling bodies. In such cases the quantities, though unlike in kind, have a mutual dependence.

459. Since the antecedent is a dividend and the consequent a divisor, any change in either or both terms will be governed by the general principles of division (§ 169). We have only to substitute the terms *antecedent*, *consequent*, and *ratio*, for *dividend*, *divisor*, and *quotient*, and these principles become simple.

PRINCIPLES.—I. *Multiplying the antecedent multiplies the ratio; dividing the antecedent divides the ratio.*

II. *Multiplying the consequent divides the ratio; dividing the consequent multiplies the ratio.*

III. *Multiplying or dividing both antecedent and consequent by the same number does not alter the ratio.*

These three principles may be embraced in one law.

GENERAL LAW.—*A change in the antecedent produces a* LIKE *change in the ratio; but a change in the consequent produces an* OPPOSITE *change in the ratio.*

460. Since the *ratio* of two numbers is equal to the antecedent divided by the consequent, it follows that

PRINCIPLE.—*The antecedent is equal to the consequent multiplied by the ratio, and the consequent is equal to the antecedent divided by the ratio.*

Examples.

461. What is the ratio of:

1. $80 : 120$? 3. $26 : \frac{7}{13}$? 5. $\frac{7}{10} : \frac{1}{5}$?

2. $60 : 8\frac{4}{7}$? 4. $2\frac{1}{2} : 7\frac{1}{5}$? 6. 120 rd. : 1 mi.?

7. Find the reciprocal of the ratio of 42 to 28.

Find the missing term:

8. Ant. 15 Ratio $\frac{4}{5}$. 10. Cons. $6.12\frac{1}{2}$ Ratio 25.

9. Cons. $3\frac{1}{4}$ Ratio 7. 11. Ant. $\frac{3}{8}$ Ratio $\frac{1}{6}$.

12. If the antecedent is $\frac{1}{3}$ of $\frac{5}{8}$ and the consequent .75, what is the ratio?

PROPORTION.

462. **Proportion** is a comparison of equal ratios. Thus, the ratios 10 : 5 and 12 : 6, each being equal to 2, may form a proportion.

463. Proportion is indicated in three ways:

1. By a double colon placed between the two ratios; thus, 3 : 4 : : 9 : 12 expresses the proportion between the numbers 3, 4, 9, and 12, and is read, 3 is to 4 as 9 is to 12.

2. By the sign of equality placed between two ratios; thus, 3 : 4 = 9 : 12 expresses proportion, and may be read as above, or, the ratio of 3 to 4 equals the ratio of 9 to 12.

3. By employing the fractional method of indicating ratio; thus, $\frac{3}{4} = \frac{9}{12}$ indicates proportion, and may be read like either of the above forms.

464. Since each ratio consists of two terms, every proportion must consist of at least four terms.

NOTE.—Each ratio is called a *couplet* and each term a *proportional*.

465. The **Antecedents** of a proportion are the first and third terms, that is, the antecedents of its ratios. The **Consequents** of a proportion are the second and fourth terms, or the consequents of its ratios.

466. The **Extremes** are the first and fourth terms. The **Means** are the second and third terms.

467. *Three* numbers are proportional when the first is to the second as the second is to the third. Thus, the numbers 4, 6, and 9 are proportional, since 4 : 6 : : 6 : 9, the ratio of each couplet being $\frac{2}{3}$.

468. In a proportion in which the means are equal, either mean is called a **Mean Proportional** between the first and fourth

terms. Thus, in 4 : 6 : : 6 : 9, 6 is a mean proportional between 4 and 9.

469. From the nature of ratio and proportion we derive the following principles:

PRINCIPLES. — I. *In every proportion the product of the means is equal to the product of the extremes.*

For in 15 : 3 : : 20 : 4, $\frac{15}{3} = \frac{20}{4}$. Reducing the fractions to a common denominator, $\frac{15 \times 4}{12} = \frac{20 \times 3}{12}$. Since these two equal fractions have the same denominator, the numerator of the first, which is the product of the extremes, must be equal to the numerator of the second, which is the product of the means.

II. *The square of a mean proportional is equal to the product of the other two terms.*

For, in the proportion 4 : 6 : : 6 : 9 since the product of the means equals the product of the extremes, $6^2 = 4 \times 9$, and 6^2 is the square of the mean proportional.

III. *Four numbers that are proportional in the direct order are proportional by inversion, and also by alternation, or by inverting the means.*

Thus, the proportion 2 : 3 : : 6 : 9, by *inversion* becomes 3 : 2 : : 9 : 6, and by *alternation* 2 : 6 : : 3 : 9.

470. Since in every proportion the product of the means equals the product of the extremes, it follows that, any three terms of a proportion being given, the fourth may be found by the following rule:

RULE. — I. *Divide the product of the extremes by one of the means, and the quotient will be the other mean.*

II. *Divide the product of the means by one of the extremes, and the quotient will be the other extreme.*

Thus, in the proportion 2 : 3 : : ? : 9, $\frac{2 \times 9}{3}$, the product of the extremes divided by the mean gives us 6, the missing mean. In 2 : 3 : : 6 : ?, $\frac{3 \times 6}{2}$, the product of the means divided by the extreme gives us 9, the missing extreme.

Examples.

471. The required term in an operation will be denoted by ?, which may be read "what."

Find the term not given in each of the following proportions:

1. $4 : 26 :: 10 : ?$.
2. $48 : 20 :: ? : 50$.
3. $42 : 70 :: 3 : ?$.
4. $? : 300 :: 20 : 100$.
5. $1 : ? :: 7 : 84$.
6. $\frac{1}{4} : \frac{1}{8} :: \frac{1}{2} : ?$.
7. $4\frac{1}{4} : 38\frac{1}{4} :: ? : 76\frac{1}{2}$.
8. \$8865 : \$720 : : ? : 16 A.
9. $4\frac{1}{2}$ yd. : ? : : \$9.75 : \$29.25.
10. ? : 21 A. 320 sq. rd. : : \$1260 : \$750.
11. 7.50 : 18 : : ? : $7\frac{1}{16}$ oz.
12. 7 oz. : ? : : £30 : £407 2*s.* $10\frac{3}{7}$*d.*
13. ? : 15 hhd. : : \$2.39 : \$.3585.
14. 1 T. 7 cwt. 3 qr. 20 lb. : 13 T. 5 cwt. 2 qr. : : \$9.50 : ?.
15. \$175.35 : ? : : $\frac{1}{8}$: $\frac{2}{7}$.
16. ? : \$$12\frac{1}{2}$: : $240\frac{1}{7}$: $149\frac{177}{1127}$.
17. $\frac{4}{5}$ yd. : ? : : \$$\frac{7}{8}$: \$59.0625.

Cause and Effect.

472. Every question in proportion may be considered as a comparison of two *causes* and two *effects*. Thus, if 3 dollars as a *cause* will buy 12 pounds as an *effect*, 6 dollars as a *cause* will buy 24 pounds as an *effect*. Or, if 5 horses as a *cause* consume 10 tons as an *effect*, 15 horses as a *cause* will consume 30 tons as an *effect*.

NOTE. — It must be borne in mind that cause and effect is not proportion, but in many cases this form of reasoning furnishes the easiest means for reaching a correct statement of proportions.

Causes may be regarded as *action*, of whatever kind, the producer, the consumer, men, animals, time, distance, weight, goods bought or sold, money at interest, etc.

Effects may be regarded as whatever is accomplished by action of any kind, the thing produced or consumed, money paid, etc.

Causes and effects in proportion are of two kinds — simple and compound.

473. A **Simple Cause** or **Effect** contains but one element; as price, quantity, cost, time, distance, or any single factor used as a term in proportion.

474. A **Compound Cause** or **Effect** is the product of two or more elements; as the number of workmen taken in connection with the time employed, length taken in connection with breadth and depth, capital considered with reference to the time employed, etc.

475. Causes and effects that admit of computation, that is, involve the *idea of quantity*, may be represented by numbers, which will have the same relation to each other as the things they represent. And since it is a principle of philosophy that *like causes produce like effects*, and that *effects are always in proportion to their causes*, we have the following proportions:

1st Cause : 2d Cause : : 1st Effect : 2d Effect,
Or, 1st Effect : 2d Effect : : 1st Cause : 2d Cause.

The two causes, or effects, forming one couplet, must be *like* numbers, and of the same denomination.

Considering all the terms of the proportion as *abstract numbers*, we may say that

1st Cause : 1st Effect : : 2d Cause : 2d Effect;

which will produce the same numerical result.

But as ratio is the result of comparing two numbers or things of the *same kind* (§ 450), the first form is regarded as more natural and philosophical.

476. Simple causes and simple effects give rise to simple ratios; compound causes and compound effects to compound *ratios.*

SIMPLE PROPORTION.

477. Simple Proportion is an equality of two simple ratios, and consists of four terms, any three of which being given, the fourth may readily be found.

NOTES. — 1. When three terms of a proportion are given, the method of finding the fourth is called the *Rule of Three*.

2. Questions in simple proportion involve only simple causes and simple effects.

Examples.

478. To find the missing term. — First method.

1. If \$8 will buy 36 yards of velveteen, how many yards may be bought for \$12?

STATEMENT.

\$		\$		yd.		yd.
8	:	12	::	36	:	?
1st cause.		2d cause.		1st effect.		2d effect.

OPERATION.

$$8 \times ? = 12 \times 36$$

$$? = \frac{\overset{6}{\cancel{12}} \times \overset{9}{\cancel{36}}}{\cancel{8}} = 54 \text{ yd. } Ans.$$

SOLUTION. — The required term in this example is an effect; and the statement is, \$8 is to \$12 as 36 yards is to ?, or how many yards. Dividing 12×36, the product of the means, by 8, the given extreme, we have 54 yards, the required term (§ 470).

2. If 6 horses will draw 10 tons, how many horses will draw 15 tons?

STATEMENT.

horses.		horses.		tons.		tons.
6	:	?	::	10	:	15
1st cause.		2d cause.		1st effect.		2d effect.

OPERATION.

$$\frac{\overset{3}{\cancel{15}} \times \overset{3}{\cancel{6}}}{\cancel{10}} = 9 \text{ horses } Ans.$$

Or,

$$\begin{array}{r|l} \cancel{10} & \cancel{15}\,3 \\ ? & \cancel{6}\,3 \\ \hline ? & = 9 \text{ horses } Ans. \end{array}$$

SOLUTION. — In this example a cause is required; and the statement is, 6 horses is to ?, or how many horses, as 10 tons is to 15 tons. Dividing 15×6, the product of the extremes, by 10, the given mean, we have 9, the required term (§ 470).

RULE.—I. *Arrange the terms in the statement so that the causes shall compose one couplet, and the effects the other, putting ? in the place of the required term.*

II. *If the required term is an extreme, divide the product of the means by the given extreme; if the required term is a mean, divide the product of the extremes by the given mean.*

NOTES.—1. If the terms of any couplet are of different denominations, they must be reduced to the same unit value.

2. If the odd term is a compound number, it must be reduced either to its lowest unit, or to a fraction or a decimal of its highest unit.

3. If the divisor and dividend contain one or more factors common to both, they should be canceled. If any of the terms of a proportion contain mixed numbers, they should first be changed to improper fractions, or the fractional part to a decimal.

4. When the vertical line is used, the divisor and ? are written on the left, and the factors of the dividend on the right.

479. There is another method of solving questions in proportion which depends more strictly upon the principles of ratio and proportion, and hence is regarded by many as simpler and more logical.

Every question in simple proportion gives *three* terms to find a *fourth.*

Of the three given numbers, one must always be of the same denomination as the required number.

The remaining two will be *like* numbers, and will bear the same relation to each other that the third does to the required number; in other words, the ratio of the third to the required number will be the same as the ratio of the other two numbers.

Regarding the third or odd term as the antecedent of the second couplet of a proportion, we find the consequent or required term by dividing the antecedent by the ratio (§ 460).

By comparing the two *like* numbers, in any given question, with the third, we may readily determine whether the answer, or required term, will be *greater* or *less* than the third term. If *greater*, then the ratio will be *less* than 1, and the two *like* numbers may be arranged in the form of a proper fraction as a divisor; if the answer, or required term, is to be *less* than the third term, then the ratio will be *greater* than 1, and the *two like* numbers may be arranged in the form of an improper *fraction*, as a divisor.

480. To find the missing term. — Second method.

1. If 4 cords of wood cost $12, how much will 20 cords cost?

OPERATION.

$4 : 20 : : 12 : ?$

$$12 \div \frac{4}{20} = \frac{\overset{3}{\cancel{12}} \times 20}{\cancel{4}} = \$\,60\ \textit{Ans.}$$

$$\text{Or,}\qquad \frac{20 \times \overset{3}{\cancel{12}}}{\cancel{4}} = \$\,60\ \textit{Ans.}$$

SOLUTION. — It will be readily seen in this example, that 4 cords and 20 cords are the *like* terms, and that $12 is the third term, and of the same denomination as the answer or required term.

If 4 cords cost $ 12, will 20 cords cost more, or less? Evidently more: then the answer or required term will be *greater* than the third term, and the ratio *less* than 1. The ratio of 4 cords to 20 cords is $\frac{4}{20}$, or $\frac{1}{5}$; hence the ratio of $ 12 to the answer must be $\frac{1}{5}$, and the answer will be 5 times $ 12, which is $ 60.

Or, after having stated the proportion 4 : 20 : : 12 : ?, we find the product of the means and divide by the extreme (§ 470).

2. If 12 yd. of cloth cost $ 48, how much will 4 yd. cost?

OPERATION.

$12 : 4 : : 48 : ?$

$$48 \div \frac{12}{4} = \overset{4}{\cancel{48}} \times \frac{4}{\cancel{12}} = \$\,16\ \textit{Ans.}$$

$$\text{Or,}\qquad \frac{4 \times \overset{4}{\cancel{48}}}{\cancel{12}} = \$\,16\ \textit{Ans.}$$

SOLUTION. — In this example we see that 12 yards and 4 yards are the *like* terms, and $ 48 the third term, and of the same denomination as the required answer.

If 12 yards cost $ 48, will 4 yards cost more or less? Less: then the ratio will be *greater* than 1, and the divisor an improper fraction. The ratio of 12 yards to 4 yards is 3, hence the ratio of $ 48 to the answer is 3, and the answer will be $\frac{1}{3}$ of $ 48, which is $ 16. Or, dividing the product of the means by the extreme (§ 470), we have $4 \times 48 \div 12 = \$\,16$.

RULE. — I. *With the two given numbers, which are of the same name or kind, form a ratio greater or less than* 1, *according as the answer is to be less or greater than the third given number.*

II. *Divide the third number by this ratio, and the quotient will be the required number or answer.*

1. Mixed numbers should first be reduced to improper fractions, and the ratio of the fractions found according to § 458.

2. *Reductions* and cancellation may be applied as in the first method.

The following examples may be solved by either method.

3. If 48 cords of wood cost \$120, how much will 20 cords cost?

4. If 6 bushels of corn cost \$4.75, how much will 75 bushels cost?

5. If 12 horses consume 42 bushels of oats in 3 weeks, how many bushels will 20 horses consume in the same time?

6. If 9 bushels of wheat make 2 barrels of flour, how many barrels of flour will 100 bushels make?

7. If 18 bushels of wheat are bought for \$22.25, and sold for \$26.75, how much will be gained on 240 bushels, at the same rate of profit?

8. If $6\frac{1}{2}$ bushels of oats cost \$3, how much will $9\frac{1}{4}$ bushels cost?

9. How much will 87.5 yards of ribbon cost, if $1\frac{3}{4}$ yards cost \$.42?

10. If by selling \$1500 worth of dry goods I gain \$275.40, what amount must I sell to gain \$1000?

11. If 20 men can perform a piece of work in 15 days, how many men must be added to the number, that the work may be accomplished in $\frac{4}{5}$ of the time?

12. If 100 yd. of broadcloth cost \$ $473.07\frac{9}{12}$, how much will 3.25 yd. cost?

13. If 1 lb. 4 oz. 10 pwt. of gold may be bought for \$260.70, how much may be bought for \$39.50?

14. In what time can a man pump 54 barrels of water, if he pumps 24 barrels in 1 h. 14 min.?

15. If $\frac{4}{5}$ of a bushel of peaches costs \$ $\frac{12}{25}$, what part of a bushel can be bought for \$ $\frac{7}{20}$?

16. If a man gains \$1870.65 by his business in 1 yr. 3 mo., how much will he gain at that rate in 2 yr. 8 mo.?

17. Two numbers are to each other as 5 to $7\frac{1}{2}$, and the less is 164.5; what is the greater?

18. If the moon moves $13° 10' 35''$ in 1 day, in what time will it perform one revolution?

[redacted]

19. If I borrow $500, and keep it 1 yr. 4 mo., for how long a time should I lend $240 as an equivalent for the favor?

20. How many days will 12 men require to do a piece of work that 95 men can do in $7\frac{1}{2}$ days?

21. If a staff 4 ft. long casts a shadow 7 ft. in length, what is the height of a tower that casts a shadow of 198 ft. at the same time?

22. A man failing in business owes $972, and his entire property is worth but $607.50. How much will a creditor receive on a debt of 11.33\frac{1}{3}$?

23. A man can perform a certain piece of work in 18 days by working 8 hours a day. In how many days can he do the same work by working 10 hours a day?

24. How much land worth $16.50 an acre, should be given in exchange for 140 acres, worth $24.75 an acre?

25. What number of men must be employed to finish a piece of work in 5 days, which 15 men could do in 20 days?

26. At $1.58 per oz., what will be the cost of a service of silver plate weighing 15 lb. 11 oz. 13 pwt. 17 gr.?

27. A borrows $1200, and keeps it 2 yr. 5 mo. 5 da. What sum should he lend for 1 yr. 8 mo. to balance the favor?

28. A farmer has hay worth $9 a ton, and a merchant has flour worth $5 per barrel. If in trading the former asks $10.50 for his hay, how much should the merchant ask for his flour?

29. A cistern holding 20 barrels has two pipes, by one of which it receives 120 gallons in an hour, and by the other discharges 80 gallons in the same time. In how many hours will it be filled?

30. A merchant in selling groceries sells $14\frac{9}{16}$ oz. for a pound. How much does he cheat a customer who buys of him to the amount of $38.40?

31. B and C have each a farm; B's farm is worth $32.50 an acre, and C's $28.75; but in trading B values his at $40 an acre. What value should C put upon his?

COMPOUND PROPORTION.

481. Compound Proportion is an expression of equality between a compound and a simple ratio, or between two compound ratios.

It embraces the class of questions in which the causes, or the effects, or both, are compound. The required term must be either a simple cause or effect, or a single element of a compound cause or effect.

Examples.

482. To find the missing term. — First method.

1. If 16 horses consume 128 bu. of oats in 50 days, how many bushels will 5 horses consume in 90 days?

STATEMENT.

1st cause.		2d cause.		1st effect.		2d effect.
$\left\{\begin{matrix}16\\50\end{matrix}\right.$	:	$\left\{\begin{matrix}5\\90\end{matrix}\right.$	::	128	:	?

Or, $16 \times 50 : 5 \times 90 :: 128 : ?$

OPERATION.

$$\frac{\cancel{5}\times\overset{9}{\cancel{90}}\times\overset{8}{\cancel{128}}}{\cancel{16}\times\underset{5}{\cancel{50}}} = 72 \text{ bu. } \textit{Ans.}$$

SOLUTION. — In this example the required term is the second effect; and the question may be read: If 16 horses in 50 days consume 128 bushels of oats, 5 horses in 90 days will consume how many bushels? Dividing the product of the means by the extremes, we have 72 bu.

NOTE. — These examples are most readily performed by cancellation.

2. If $480 gains $84 interest in 30 months, what sum will gain $21 in 15 months?

STATEMENT.

1st cause.		2d cause.		1st effect.		2d effect.
$\left\{\begin{matrix}480\\30\end{matrix}\right.$	:	$\left\{\begin{matrix}?\\15\end{matrix}\right.$	::	84	:	21

OPERATION.

$$\frac{\overset{120}{\cancel{480}}\times\overset{2}{\cancel{30}}\times 21}{\underset{4}{\cancel{84}}\times\cancel{15}} = \$240 \ \textit{Ans.}$$

SOLUTION. — The required term in this example is an element of the second cause; and the question may be read: If $480 in 30 months gains $84, what principal in 15 months will gain $21? Dividing the product of the extremes by the means, we have $240.

3. If 7 men dig a ditch 60 feet long, 8 feet wide, and 6 feet deep in 12 days, what length of ditch can 21 men dig in $2\frac{2}{3}$ days, if it is 3 feet wide and 8 feet deep?

STATEMENT.

$$\underset{\text{1st cause.}}{\begin{cases}7\\12\end{cases}} : \underset{\text{2d cause.}}{\begin{cases}21\\2\frac{2}{3}\end{cases}} :: \underset{\text{1st effect.}}{\begin{cases}60\\8\\6\end{cases}} : \underset{\text{2d effect.}}{\begin{cases}?\\3\\8\end{cases}}$$

Or, $$7 \times 12 : 21 \times \tfrac{8}{3} :: 60 \times 8 \times 6 : ? \times 3 \times 8.$$

OPERATION.

$$\frac{\overset{7}{\cancel{21}} \times 8 \times \overset{5}{\cancel{60}} \times \cancel{8} \times \overset{2}{\cancel{6}}}{\cancel{7} \times \cancel{3} \times \cancel{12} \times \cancel{3} \times \cancel{8}} = 80 \text{ ft. } \textit{Ans.}$$

Or,

$$\begin{array}{r|l} & \cancel{7} \\ \cancel{7} & \cancel{21} \\ \cancel{3} & 8 \\ \cancel{12} & \overset{5}{\cancel{60}} \\ \cancel{3} & \cancel{8} \\ \cancel{8} & \overset{2}{\cancel{6}} \\ ? & \end{array}$$

$$? = 8 \times 5 \times 2 = 80 \text{ ft. } \textit{Ans.}$$

SOLUTION.—In this example the required term is the *length* of the ditch, and is an element of the second effect. The question, as stated, will read thus: if 7 men, in 12 days, dig a ditch 60 feet long, 8 feet wide, and 6 feet deep, 21 men, in $2\frac{2}{3}$ days, will dig a ditch how many feet long, 3 feet wide, and 8 feet deep? Dividing the product of the means by the extreme (§ 470), we have 80 ft.

RULE.—I. *Of the given terms, select those which constitute the various causes, and those which constitute the various effects, and arrange them in couplets, putting an interrogation point in place of the required term.*

II. *Then, if the unknown term expressed by the interrogation point occurs in either of the extremes, make the product of the means a dividend, and the product of the extremes a divisor; but if the unknown term occurs in either of the means, make the product of the extremes a dividend, and the product of the means a divisor.*

NOTE.—The causes must be exactly alike in the *number* and *kind* of their terms; the same is true of the effects. The same preparation of the terms by reduction is to be observed as in simple proportion.

483. To find the missing term. — Second method.

Examples in Compound Proportion may also be solved by the Second Method in Simple Proportion (§ 480).

1. If 18 men can build 42 rods of wall in 16 days, how many men can build 28 rods in 8 days?

OPERATION.

$$\left.\begin{matrix}42 : 28 \\ 8 : 16\end{matrix}\right\} :: 18 : ?$$

$$18 \div \frac{42}{28} \times \frac{8}{16} = \overset{6}{\cancel{18}} \times \frac{\overset{2}{\cancel{28}}}{\underset{\cancel{14}}{\cancel{42}}} \times \frac{\overset{2}{\cancel{16}}}{\cancel{8}} = 24 \text{ men} \quad \textit{Ans.}$$

Or,

$$\frac{\overset{2}{\cancel{28}} \times \overset{2}{\cancel{16}} \times \overset{6}{\cancel{18}}}{\underset{\cancel{14}}{\cancel{42}} \times \cancel{8}} = 24 \text{ men} \quad \textit{Ans.}$$

SOLUTION. — In this example all the terms appear in couplets, except one, which is 18 men, and that is of the same kind as the required answer.

Since compound proportion is made up of two or more simple proportions, if this third or odd term is divided by the compound ratio, or by the simple ratio of each couplet successively, the product will be the required term.

By comparing the terms of each couplet with the third term, we may readily determine whether the answer, or term sought, will be greater or less than the third term; if *greater*, then the ratio will be *less* than 1, and the divisor a proper fraction; if *less*, the ratio will be *more* than 1, and the divisor an improper fraction.

First, we compare the terms composing the first couplet, 42 rods and 28 rods, with the third term, 18 men. If 42 rods require 18 men, how many men will 28 rods require? Less men; hence the ratio is *greater* than 1, and the divisor an improper fraction, $\frac{42}{28}$. Next, if 16 days require 18 men, how many men will 8 days require? More men; hence the ratio is *less* than 1, and the divisor a proper fraction, $\frac{8}{16}$. Regarding the third term as the antecedent of a couplet, the consequent being the term sought, if we divide this third term by the simple ratios, or by their product, we shall have the required term or answer, thus: $18 \div \frac{42}{28} \times \frac{8}{16} = 24$, as shown in the operation.

Or, finding the product of the means and dividing by the extremes, we have $\frac{28 \times 16 \times 18}{42 \times 8} = 24$.

2. If 5 compositors, in 16 days of 14 hours each, can compose 20 sheets of 24 pages in each sheet, 50 lines in a page, and 40 letters in a line, in how many days, of 7 hours each, will 10 compositors compose a volume to be printed in the same letter, containing 40 sheets, 16 pages in a sheet, 60 lines in a page, and 50 letters in a line?

OPERATION.

Days. Comp. Hours. Sheets. Pages. Lines. Letters.

$$16 \div \tfrac{10}{5} \times \tfrac{7}{14} \times \tfrac{20}{40} \times \tfrac{24}{16} \times \tfrac{50}{60} \times \tfrac{40}{50} = 32 \text{ days } \textit{Ans.}$$

BY CANCELLATION.

$$\begin{array}{r|l} & 16 \\ {}^{2}\cancel{10} & \cancel{5} \\ \cancel{7} & \cancel{14}^{2} \\ \cancel{20} & \cancel{40} \\ {}^{3}\cancel{24} & \cancel{16}^{2} \\ \cancel{50} & \cancel{60}^{3} \\ \cancel{40} & 50 \\ \hline \end{array}$$

32 days *Ans.*

SOLUTION. — The required term or answer is to be in *days*; and we see that all the terms appear in pairs or couplets, except the 16 days, which is of the same kind as the answer sought.

We proceed to compare the terms of each couplet with the 16 days. First, if 5 compositors require 16 days, how many days will 10 compositors require? Less days; hence the divisor is the improper fraction $\frac{10}{5}$, and we have $16 \div \frac{10}{5}$. Next, if 14 hours a day require 16 days, how many days will 7 hours a day require? More days; hence the divisor is the proper fraction $\frac{7}{14}$, and we have $16 \div \frac{10}{5} \times \frac{7}{14}$. Next, if 20 sheets require 16 days, how many days will 40 sheets require? More days; hence the divisor is the proper fraction $\frac{20}{40}$, and we have $16 \div \frac{10}{5} \times \frac{7}{14} \times \frac{20}{40}$. Pursuing the same method with the other couplets, we obtain the result as shown in the operation.

RULE. — I. *Of the terms composing each couplet form a ratio greater or less than* 1, *in the same manner as if the answer depended on those two and the third or odd term.*

II. *Divide the third or odd term by these ratios successively, and the quotient will be the answer sought.*

NOTE. — By the *odd* term is meant the one that is of the same kind as the answer.

The following examples may be solved by either method:

3. If 12 horses plow 11 acres in 5 days, how many horses would plow 33 acres in 18 days?

4. If 480 bushels of oats will last 24 horses 40 days, how long will 300 bushels last 48 horses, at the same rate?

5. If 7 reaping machines can cut 1260 acres in 12 days, in how many days can 16 machines reap 4728 acres?

6. If 144 men in 6 days of 12 hours each, build a wall 200 ft. long, 3 ft. high, and 2 ft. thick, in how many days of 7 hours each can 30 men build a wall 350 ft. long, 6 ft. high, and 3 ft. thick?

7. In how many days will 6 persons consume 5 bu. of potatoes, if 3 bu. 3 pk. last 9 persons 22 days?

8. If the use of $ 300 for 1 yr. 8 mo. is worth $ 30, how much is the use of $ 210.25 for 3 yr. 4 mo. 24 da. worth?

9. If a cistern 16 ft. long, 7 ft. wide, and 15 ft. deep cost $ 36.72, how much, at the same rate per cu. ft., would another cistern cost that is $17\frac{1}{2}$ ft. long, $10\frac{1}{2}$ ft. wide, and 16 ft. deep?

10. If 468 bricks, 8 inches long and 4 inches wide, are required for a walk 26 ft. long and 4 ft. wide, how many bricks will be required for a walk 120 ft. long and 6 ft. wide?

11. If a cistern $17\frac{1}{2}$ ft. long, $10\frac{1}{2}$ ft. wide, and 13 feet deep holds 546 barrels, how many barrels will a cistern hold that is 16 ft. long, 7 ft. wide, and 15 ft. deep?

12. If 11 men can cut 147 cords of wood in 7 days, when they work 14 hours per day, how many days will it take 5 men to cut 150 cords, working 10 hours each day?

13. If it costs $ 95.60 to carpet a room 24 ft. by 18 ft., how much will it cost to carpet a room 38 ft. by 22 ft. with the same material?

14. A miller has a bin 8 ft. long, $4\frac{1}{5}$ ft. wide, and $2\frac{1}{2}$ ft. deep, and its capacity is 75 bu.; how deep must he make another bin which is to be 18 ft. long and $3\frac{5}{6}$ feet wide, that its capacity may be 450 bu.?

15. If 4 men in $2\frac{1}{2}$ days, mow $6\frac{3}{8}$ acres of grass, by working $8\frac{1}{4}$ hours a day, how many acres will 15 men mow in $3\frac{3}{4}$ days, by working 9 hours a day?

16. If 248 men, in $5\frac{1}{2}$ days of 12 hours each, dig a ditch of 7 degrees of hardness, $232\frac{1}{2}$ yd. long, $2\frac{3}{8}$ yd. wide, and $2\frac{1}{8}$ yd. deep; in how many days of 9 hours each will 24 men dig a ditch of 4 degrees of hardness, $387\frac{1}{2}$ yd. long, $5\frac{1}{4}$ yd. wide, and $3\frac{1}{2}$ yd. deep?

PARTITIVE PROPORTION.

484. Partitive Proportion is the process of dividing a number into parts which bear a given relation to each other.

Examples.

485. To divide a number into proportional parts.

1. Divide \$ 1305 into parts proportional to the numbers 2, 3, and 4.

OPERATION.

$2+3+4=9$

$\left.\begin{array}{l}\frac{2}{9} \text{ of } \$1305 = \$290 \\ \frac{3}{9} \text{ of } \$1305 = \$435 \\ \frac{4}{9} \text{ of } \$1305 = \$580\end{array}\right\} Ans.$

SOLUTION.—The problem is to take three fractions of \$1305 which shall be in the proportion of 2, 3, 4. Any fractions with a common denominator, and with 2, 3, and 4 as respective numerators, would be in the proper proportion; but since the sum of the three parts of \$1305, when found, must equal the *whole* of \$1305, the sum of the three fractions must equal a *whole unit*, and since the sum of the numerators is $2+3+4=9$, the common denominator must be 9, since $\frac{9}{9}=1$. Therefore the fractions must be $\frac{2}{9}$, $\frac{3}{9}$, $\frac{4}{9}$. $\frac{2}{9}$ of \$ 1305 = \$ 290; $\frac{3}{9}$ of \$ 1305 = \$ 435; $\frac{4}{9}$ of \$ 1305 = \$ 580.

2. A, B, and C leased a store for \$ 3300 a year. A's space costs twice as much as B's, and C's costs four times as much as A's. What is the yearly rental for each?

OPERATION.

$1+2+8=11$

$\left.\begin{array}{l}\frac{1}{11} \text{ of } \$3300 = \$300, \text{ B's rental.} \\ \frac{2}{11} \text{ of } \$3300 = \$600, \text{ A's } \quad " \\ \frac{8}{11} \text{ of } \$3300 = \$2400, \text{ C's } \quad "\end{array}\right\} Ans.$

SOLUTION.— Assuming B's rental to be 1 part, A's will be 2×1, or 2, and C's 4×2, or 8. We then have \$3300 to be divided into parts proportional to the numbers 1, 2, and 8. Proceeding as in Example 1, the answers are, \$600 for A, \$300 for B, and \$2400 for C.

3. Divide 5200 into parts proportional to $\frac{1}{2}$, $\frac{1}{5}$, and $\frac{1}{6}$.

OPERATION.

$\frac{1}{2}=\frac{15}{30}$; $\frac{1}{5}=\frac{6}{30}$; $\frac{1}{6}=\frac{5}{30}$

$15+6+5=26$

$\left.\begin{array}{l}\frac{15}{26} \text{ of } 5200 = 3000 \\ \frac{6}{26} \text{ of } 5200 = 1200 \\ \frac{5}{26} \text{ of } 5200 = 1000\end{array}\right\} Ans.$

SOLUTION.—When fractions have a common denominator, they have the ratios of their numerators (§ 458). Reducing $\frac{1}{2}$, $\frac{1}{5}$, and $\frac{1}{6}$ to common denominators, we have $\frac{15}{30}$, $\frac{6}{30}$, and $\frac{5}{30}$. The problem now is to divide 5200 into parts proportional to 15, 6, 5. Proceeding as in Example 1, we have $15+6+5=26$. $\frac{15}{26}$ of $5200 = 3000$; $\frac{6}{26}$ of $5200 = 1200$; $\frac{5}{26}$ of $5200 = 1000$.

RULE. — I. *Find the sum of the numbers in proportion for a common denominator. Form fractions with this sum as denominator, and the proportional numbers as respective numerators. Take these fractional parts of the number given.*

II. *When the numbers in proportion are fractions, reduce them to a common denominator. The numerators may then be regarded as whole numbers in the given proportion, with which proceed as before.*

4. Divide 105 into parts proportional to 5, 7, 9.

5. Divide 126 into six parts proportional to 1, 2, 3, 4, 5, 6.

6. Divide 666 into three parts proportional to $\frac{1}{4}$, $\frac{1}{6}$, and $\frac{1}{12}$.

7. A man bequeathed $ 63000 to three heirs as follows: To his son $\frac{1}{3}$ as much as to his wife, and to his daughter $\frac{1}{2}$ as much as to his son. How much did each get?

8. A man bought three houses for $ 112500. The first cost twice as much as the second, and the third three times as much as the first. How much did he pay for each?

9. A, B, and C, owned a vessel worth $ 52000. B owned $\frac{2}{3}$ as much as A, and C $\frac{1}{2}$ as much as A. How much did each own?

10. Four persons rent a farm of 115 A. 32 sq. rd. at $ 3.75 an acre. A puts on 144, B 160, C 192, and D 324 sheep. How much rent ought each to pay?

11. Three persons gain $ 2640, of which B is to have $ 6 as often as C $ 4, and D $ 2. What is each one's share?

12. Six persons are to share among them $ 6300; A is to have $\frac{1}{7}$ of it, B $\frac{1}{5}$, C $\frac{2}{9}$, D is to have as much as A and C together, and the remainder is to be divided between E and F in the ratio of 3 to 5. How much does each receive?

13. A father divides his estate worth $ 5463.80 between his two sons, giving the elder $\frac{1}{8}$ more than the younger. How much is each son's share?

14. Three men, A, B, and C, enter into partnership. For every $ 3 of capital A puts in, B puts in $ 4, and C $ 1. Their whole capital amounts to $ 20480. How much money does each put in?

PARTNERSHIP.

486. A **Partnership** is an association of persons for the transaction of business.

487. The **Partners** are the individuals thus associated.

Partners may be active or silent, general, limited or special. An *active* partner takes an active part in the business, while a *silent* partner merely furnishes capital. A *general* partner risks his whole property in the business, and must forfeit it all if necessary to pay the debts. A *special* or *limited* partner is responsible only for a limited sum which is specified in the agreement.

When partners are to share in the profits and losses of the business in proportion to the amount of capital contributed, this must be distinctly stated in the agreement.

488. A **Firm, Company,** or **House** is any particular partnership association.

489. **Capital,** or **Stock,** is the money or property invested by the partners, called also **Investment,** or **Joint Stock.**

490. The **Resources** of a firm are the amounts due the firm, together with the property of all kinds belonging to it, called also **Assets,** or **Effects.**

491. The **Liabilities** of a firm are its debts.

The **Net Capital** is the excess of resources over liabilities. A **Deficit** is the excess of liabilities over resources. When there is a net capital the firm is *solvent;* but when there is a *deficit* the firm is *insolvent* or *bankrupt.*

492. **Dividends** are the profits to be divided among the partners of a company. **Assessments** are the sums to be paid by the partners to meet losses sustained.

493. Computations in partnership may be simple or *compound.*

SIMPLE PARTNERSHIP.

494. In **Simple Partnership** the capital of each partner is invested for the same time, and the profits and losses are shared in proportion to the capital contributed.

Examples.

495. To find each partner's share of the profit or loss, when the capital of all is employed for equal periods of time.

1. A and B engage in trade; A furnishes \$500, and B \$700 as capital; they gain \$96. What is each man's share of gain?

OPERATION.

$\$500 + \$700 = \$1200$, whole cap.
$\frac{500}{1200} = \frac{5}{12}$, A's part of the cap.
$\frac{700}{1200} = \frac{7}{12}$, B's " " " "
$\frac{5}{12}$ of $\$96 = \$40$, A's gain } *Ans.*
$\frac{7}{12}$ of $\$96 = \$56$, B's " } *Ans.*

SOLUTION. — The whole amount of capital employed is $\$500 + \$700 = \$1200$; hence, A furnishes $\frac{500}{1200} = \frac{5}{12}$ of the capital, and B furnishes $\frac{700}{1200} = \frac{7}{12}$ of the capital. Since each man's share of the profit will have the same ratio to the whole profit as his part of the capital has to the whole capital, A will have $\frac{5}{12}$ of the \$96, and B $\frac{7}{12}$ of the \$96, for their respective shares of the profits.

We may also regard the whole capital as the *first cause*, and each man's share of the capital as the *second cause*, the whole profit or loss as the *first effect*, and each man's share of the profit or loss as the *second effect*, and solve by proportion thus:

1st cause.		2d cause.		1st effect.		2d effect.		
\$1200	:	\$500	::	\$96	:	?	= \$40, A's gain	*Ans.*
\$1200	:	\$700	::	\$96	:	?	= \$56, B's "	*Ans.*

RULE. — *Multiply the whole profit or loss by the ratio of each man's share of the capital to the whole capital.* Or,

Solve by proportion. The whole capital is to each man's share of the capital as the whole profit or loss is to each man's share of the profit or loss.

2. Three men trade in company; A furnishes \$8000, B \$12000, and C \$20000 of the capital; their gain is \$1800. *What is each* man's share?

3. Three men engage in trade; A puts in $ 6470, B $ 3780, C $ 9860, and they gain $ 7890. What is each partner's share of the profit?

4. B and C buy pork to the amount of $ 1847.50, of which B pays $ 739, and C the remainder; they gain $ 375. What is each one's share of the gain?

5. A, B, and C form a company for the manufacture of woolen cloths. A puts in $ 10000, B $ 12800, and C $ 3200. C is allowed $ 1500 a year for personal attention to the business; their expenses for labor, clerk hire, and other incidentals for 1 year are $ 3400, and their receipts during the same time amount to $ 9400. What is each one's income from the business?

6. Three men trade in company. A furnishes $ 8000, and B $ 12000. Their gain is $ 1680, of which C's share is $ 840. What are C's stock, and A's and B's gain.

7. Four persons engage in the lumber trade, and invest jointly $ 22500; at the expiration of a certain time, A's share of the gain is $ 2000, B's $ 2800.75, C's $ 1685.25, and D's $ 1014. How much capital did each put in?

8. Three persons enter into partnership for the manufacture of coal oil, with a joint capital of $ 18840. A puts in $ 3 as often as B puts in $ 5, and as often as C puts in $ 7. Their annual gain is equal to C's stock. How much is each partner's gain?

9. Three persons engaged in the lumber trade; two of the persons furnished the capital, and the third managed the business; they gained $ 2571.24, of which C received $ 6 as often as D $ 4, and E had $\frac{1}{5}$ as much as the other two for taking care of the business. How much was each one's share of the gain?

10. A cargo of corn valued at $ 3475.60 was entirely lost; $\frac{1}{3}$ of it belonged to A, $\frac{1}{4}$ of it to B, and the remainder to C. How much was the loss of each if there was an insurance of *$ 2512?*

COMPOUND PARTNERSHIP.

496. In **Compound Partnership** the time for which the capital of each partner is invested must be taken into account.

It is evident that the respective shares of profit and loss will depend equally upon two conditions, viz.: *the amount of capital* invested by each, and the *time* it is employed. Hence they will be proportional to the *products* of these two elements.

Examples.

497. To find each partner's share of the profit or loss when their capital is employed for unequal periods of time.

1. Two men form a partnership; A puts in \$ 320 for 5 months, and B \$ 400 for 6 months; they lose \$ 140. What is each man's share of the loss?

OPERATION.

$5 \times \$\,320 = \$\,1600$, A's capital for 1 mo.
$6 \times \$\,400 = \$\,2400$, B's " " "
$\$\,4000$, entire " " "

$\$\,\frac{1600}{4000} = \frac{2}{5}$, A's share in the partnership
$\$\,\frac{2400}{4000} = \frac{3}{5}$, B's " " "

$\left.\begin{array}{l}\frac{2}{5} \times \$\,140 = \$\,56, \text{ A's loss} \\ \frac{3}{5} \times \$\,140 = \$\,84, \text{ B's loss}\end{array}\right\}$ *Ans.*

SOLUTION.—The use of \$ 320 for 5 months is the same as the use of 5 times \$ 320, or \$ 1600, for one month; and the use of \$ 400 for 6 months is the same as the use of 6 times \$ 400, or \$ 2400, for 1 month; hence the use of the entire capital is the same as the use of $\$\,1600 + \$\,2400 = \$\,4000$ for 1 month. A's interest in the partnership is therefore $\frac{1600}{4000} = \frac{2}{5}$, and he will suffer $\frac{2}{5}$ of the loss; $\frac{2}{5}$ of $\$\,140 = \$\,56$ loss; and B's interest in the partnership is $\frac{2400}{4000} = \frac{3}{5}$, and he will suffer $\frac{3}{5}$ of the loss; $\frac{3}{5}$ of $\$\,140 = \$\,84$ loss.

We may also solve by proportion, the *causes* being compounded of the two elements, *capital* and *time;* thus:

$\left.\begin{array}{l}\$\,4000 : \$\,1600 :: \$\,140 : ? = \$\,56, \text{ A's loss} \\ \$\,4000 : \$\,2400 :: \$\,140 : ? = \$\,84, \text{ B's loss}\end{array}\right\}$ *Ans.*

RULE. — *Multiply each man's capital by the time it is employed in trade, and add the products. Then multiply the entire profit or loss by the ratio of each product to the sum of the products; the*

[redacted]

results will be the respective shares of profit or loss of each partner. Or,

Multiply each man's capital by the time it is employed in trade, and regard each product as his capital, and the sum of the products as the entire capital, and solve by proportion.

2. A, B, and C enter into partnership. A puts in $ 357 for 5 months, B $ 371 for 7 months, C $ 154 for 11 months, and they gain $ 347.20. How much is each one's share?

3. Three men hire a pasture for $ 55.50. A puts in 5 cows, 12 weeks; B, 4 cows, 10 weeks; and C, 6 cows, 8 weeks. How much ought each to pay?

4. B commenced business with a capital of $ 15000. Three months afterward C entered into partnership with him, and put in 125 acres of land. At the close of the year their profits were $ 4500, of which C was entitled to $1800. What was the value of the land per acre?

5. A and B engaged in trade. A put in $ 4200 at first, and 9 months afterward $ 200 more. B put in at first $ 1500, and at the end of 6 months took out $ 500. At the end of 16 months their gain was $ 772.20. What was each one's share of the gain?

6. Four companies of men worked on a railroad. In the first company there were 30 men who worked 12 days, 9 hours a day; in the second, there were 32 men who worked 15 days, 10 hours a day; in the third, there were 28 men who worked 18 days, 11 hours a day; and in the fourth, there were 20 men who worked 15 days, 12 hours a day. The entire amount paid to all the companies was $ 1500. What were the wages of each company?

7. A and B are partners. A's capital is to B's as 5 to 8; at the end of 4 months A withdraws $\frac{1}{2}$ of his capital, and B $\frac{3}{8}$ of his; at the end of the year their whole gain is $ 4000. How much belongs to each?

8. B, C, and D form a manufacturing company, with capitals of $ 15000, $ 25000, and $ 30000 respectively. After 4

months B draws out $ 1200, and in 2 months more he draws out $ 1500 more, and 4 months afterward puts in $ 1000. C draws out $ 2000 at the end of 6 months, and $ 1500 more 4 months afterward, and a month later puts in $ 800. D puts in $ 1800 at the end of 7 months, and 3 months later draws out $ 5000. If their gain at the end of 18 months is $ 15000, how much should each receive?

9. The joint stock of a company was $ 5400, which was doubled at the end of the year. A put in $\frac{1}{2}$ for $\frac{3}{4}$ of a year, B put in $\frac{2}{5}$ for $\frac{1}{2}$ of a year, and C put in the remainder for one year. What is each one's share of the entire stock at the end of the year?

10. Three men engaged in merchandising. A's money was in 10 months, for which he received $ 456 of the profits; B's was in 8 months, for which he received $ 343.20 of the profits; and C's was in 12 months, for which he received $ 750 of the profits. If their whole capital invested was $ 14345, how much was the capital of each?

11. Three men take an interest in a coal mine. B invests his capital for 4 months, and claims $\frac{1}{10}$ of the profits; C's capital is in 8 months; and D invests $ 6000 for 6 months and claims $\frac{2}{5}$ of the profits. How much did B and C put in?

12. A, B, and C engage in manufacturing shoes. A puts in $ 1920 for 6 months; B, a sum not specified for 12 months; and C, $ 1280 for a time not specified. A received $ 2400 for his stock and profits, B $ 4800 for his, and C $ 2080 for his. What were B's stock, and C's time?

13. William Gallup began trade January 1, 1895, with a capital of $ 3000, and, succeeding in business, took in M. H. Decker as a partner on the first day of the following March, with a capital of $ 2000; four months later, they admitted J. Newman as third partner, who put in $ 1800 capital; they continued their partnership until April 1, 1897, when they found that $ 4388.80 had been gained since January 1, 1895. What was each one's share?

ALLIGATION.

498. Alligation treats of mixing or compounding two or more ingredients of different values or qualities.

499. The **Mean Price** or **Value** is the average of the ingredients, or the price or value of a unit of the mixture.

500. The **Simples** are the unmixed ingredients.

501. Alligation Medial or **Average** is the process of finding the mean or average value of several ingredients (§ 186).

502. Alligation Alternate is the process of finding the quantities at given values to be used in any mixture of a given average value.

Examples.

503. To find the proportional quantity of each ingredient to be used, the mean value, and values of the ingredients, being given.

1. A farmer wishes to mix oats worth \$.50 a bushel, with peas worth \$2.00 a bushel, to make a mixture worth \$1.75 a bushel. What quantity of each may he take?

OPERATION.

$$\$1.75\left\{\begin{array}{r|c|r|} \$\ .50 & \frac{4}{5} & 4 \\ \$\,2.00 & 4 & 20 \end{array}\right.$$

SOLUTION. — If a mixture in any proportion of oats worth \$.50 a bushel, and peas worth \$2.00 a bushel, is sold for \$1.75, there will be a gain on the oats, the ingredient worth *less* than the mean price, and a loss on the peas, the ingredient worth *more* than the mean price, and if we take such quantities of each that the gain and loss are each \$1.00, a *unit of value*, the result will be the required mixture. By selling 1 bushel of oats worth \$.50 for \$1.75, there is a gain of \$1.75 – \$.50 = \$1.25, and to gain only \$1.00 we must take $\frac{100}{125} = \frac{4}{5}$ of a bu. Hence we place $\frac{4}{5}$ opposite the \$.50. By selling 1 bushel of peas worth \$2.00 for \$1.75, there is a loss of \$.25, and to lose \$1.00, we must take $\frac{100}{25} = 4$ bushels; hence we place 4 bushels

opposite the $2.00. Therefore $\frac{4}{5}$ bu. oats to 4 bu. peas are the proportional quantities required. But since multiplying both terms of a ratio by the same number does not alter its value (§ 459) we may change $\frac{4}{5}$ to an integer by multiplying both $\frac{4}{5}$ and 4 by 5, the least common multiple of the denominator; and we obtain the integers 4 and 20 for the proportional terms. That is, we may take 4 bu. of oats to 20 bu. of peas. To prove this, 4 bu. of oats at $ .50 cost $ 2.00; 20 bu. of peas at $ 2.00 cost $ 40; hence 24 bu. of the mixture cost $ 42, and 1 bu. costs $ 1.75 as required.

2. What relative quantities of candy at 7 cents, 8 cents, 11 cents, and 14 cents per pound, must a manufacturer mix to produce a mixture worth 10 cents per pound?

OPERATION.

		1	2	3	4	5
10	7	$\frac{1}{3}$		4		4
	8		$\frac{1}{2}$		1	1
	11		1		2	2
	14	$\frac{1}{4}$		3		3

Or,

		1	2	3	4	5
10	7	$\frac{1}{3}$		1		1
	8		$\frac{1}{2}$		2	2
	11	1		3		3
	14		$\frac{1}{4}$		1	1

SOLUTION. — To preserve the equality of gains and losses, we must compare two prices or simples, one *greater* and one *less* than the mean rate, and treat each pair or couplet as a separate example. Thus, comparing the simples whose prices are 7 cents and 14 cents, we find that, to gain 1 cent, $\frac{1}{3}$ of a pound at 7 cents must be taken, and, to lose 1 cent, $\frac{1}{4}$ of a pound at 14 cents must be taken; and comparing the simples the prices of which are 8 cents and 11 cents, we find that $\frac{1}{2}$ pound at 8 cents must be taken to gain 1 cent, and 1 pound at 11 cents must be taken to lose 1 cent. These proportional terms are written in columns 1 and 2. We now reduce these couplets separately to integers, as in the last example, writing the results in columns 3 and 4; and arranging all the terms in column 5, we have 4, 1, 2, and 3 for the proportional quantities required.

If we compare the prices 7 and 11 for the first couplet, and the prices 8 and 14 for the second couplet, as in the second operation, we shall obtain 1, 2, 3, and 1 for the proportional terms.

3. What quantities of flour worth $ 4, $ 5.60, and $ 6 a barrel, must be sold to realize an average price of $ 5.50 a barrel?

OPERATION.

		1	2	3	4	5
$ 5.50	4.00	$\frac{2}{3}$	$\frac{2}{3}$	2	2	4
	5.60		10		30	30
	6.00	2		6		6

SOLUTION. — Comparing the first price and the third, with $ 1 as a unit, we obtain the couplet $\frac{2}{3}$ to 2; and comparing the first price with the second we obtain the couplet $\frac{2}{3}$ to 10. Reducing the couplets to integers we find that we may take 2 barrels of the first kind with 6 of the third, and 2 of the first kind with 30 of the second. These two combinations taken together give 4 of the first kind with 30 of the second and 6 of the third.

RULE. — I. *Write the values of the several ingredients in a column, and the mean value at the left.*

II. *Consider any two values, one of which is less and the other greater than the mean value, as forming a couplet; find the difference between each of these values and the mean value, and write the reciprocal of each difference opposite the given value in the couplet, as one of the proportional terms. In like manner form the couplets, till all the values have been employed, writing each pair of proportional terms in a separate column.*

III. *If the proportional terms thus obtained are fractional, multiply each pair by the least common multiple of their denominators, and carry these integral products to a single column, adding any two or more that stand in the same horizontal line; the final results will be the proportional quantities required.*

NOTES. — 1. If the numbers in any couplet or column have a common factor, it may be rejected, and if the denominator of two or more fractions in any couplet have a common factor, it may also be rejected. Thus $\frac{1}{15} = \frac{1}{5} \times \frac{1}{3}$ and $\frac{3}{20} = \frac{1}{5} \times \frac{3}{4}$; hence $\frac{1}{5}$ may be rejected, and $\frac{1}{15} : \frac{3}{20}$ may be expressed $\frac{1}{3} : \frac{3}{4}$.

2. We may also multiply the numbers in any couplet or column by any multiplier we choose, without affecting the equality of the gains and losses, since the value of a ratio remains unchanged when both its terms are multiplied by the same number (§ 459). We may thus obtain an indefinite number of results, all of which will give correct answers.

4. A farmer had three pieces of land worth \$40, \$60, and \$80 an acre respectively. How many acres must he sell from the different tracts, to realize an average price of \$62.50 an acre?

5. A retailer places on a bargain counter silks at \$.60, \$.50, \$.42, \$.38, and \$.30 a yard, and marks them all at \$.45. How many yards of each kind must he sell so as neither to gain nor lose?

6. How may wheat worth \$.56, \$.40, and \$.60 a bushel be mixed to produce a mixture worth \$.55 a bushel?

7. What relative quantities of silver $\frac{3}{4}$ pure, $\frac{5}{6}$ pure, and $\frac{9}{10}$ pure, will make a mixture $\frac{7}{8}$ pure?

8. What relative quantities of alcohol 75%, 80%, 98%, and 99% pure, must be mixed to produce a mixture which shall be 95% pure?

504. To find the proportion of all the quantities when two or more of the quantities are required to be in a certain proportion.

1. A farmer having oats worth $ .30 per bushel, corn worth $ .52 per bushel, and wheat worth $ .62 per bushel, desires to form a mixture worth $ .50 per bushel, which shall contain equal parts of corn and wheat. In what proportion shall the ingredients be taken?

OPERATION.

		1	2	3	4	5	6
	30	$\frac{1}{20}$	$\frac{1}{20}$	1	3	6	7
50	52	$\frac{1}{2}$		10			10
	62		$\frac{1}{12}$		5	10	10

SOLUTION. — We first obtain the proportional terms in columns 3 and 4, as in Example 3, § 503, taking 1 cent as the unit or standard. Now, it is evident that the loss and gain will be equal if we take each couplet, or any multiple of each, alone; or both couplets, or any multiples of both, together. Multiplying the terms in column 4 by 2, we obtain the terms in column 5; and adding the terms in columns 3 and 5, we obtain the terms in column 6; that is, the farmer takes 7 bushels of oats to 10 of corn and 10 of wheat, which is the required proportion.

RULE. — I. *Compare the given values, and obtain the proportional term by couplets, as in* § 503.

II. *Reduce the couplets to higher or lower terms, as may be required; then select the columns at pleasure, and combine them by adding the terms in the same horizontal line, till a set of proportional terms is obtained, answering the required conditions.*

2. A grocer has four kinds of molasses, worth $ .25, $ .35, $ .28, and $ .38 per gallon, respectively. In what proportions may he mix the four kinds, to obtain a compound worth $ .30 per gallon, using equal parts of the first two kinds?

3. In what proportions may we take candy at $ .07, $ .08, $ .13, and $ .15 a pound, to form a compound worth $ .10 a pound, using equal parts of the first three kinds?

4. A miller has oats at $ .30, corn at $ .50, and wheat at $.70 per bushel. He desires to form a mixture worth $ .60 *per bushel,* using equal parts of oats and corn. What quantity *of each must* he use?

505. To find the proportion when the quantity of one of the ingredients is limited.

1. A miller has oats worth $.38, corn worth $.44, and barley worth $.65 per bushel. He wishes to form a mixture worth $.58 per bushel, containing 100 bushels of corn. How many bushels of oats and barley may he take?

OPERATION.

		1	2	3	4	5	6
58	38	$\frac{1}{20}$		7		7	700
	44		$\frac{1}{14}$		1	1	100
	65	$\frac{1}{7}$	$\frac{1}{7}$	20	2	22	2200

SOLUTION.—By § 503, we find the proportional quantities to be 7 bushels of oats to 1 of corn and 22 of barley. But as 100 bushels of corn, instead of 1, are required, we must multiply this term by 100, and to preserve the proportion the other two terms must also be multiplied by 100; we then have 700 bushels of oats, and 2200 bushels of barley.

RULE.—*Find the proportional quantities by § 503. Divide the given quantity by the proportional quantity of this ingredient, and multiply each of the other proportional quantities by the quotient thus obtained.*

2. A dairyman bought 10 cows at $20 a head. How many must he buy at $16, $18, and $24 a head, so that the whole may cost him an average price of $22 a head?

3. A man bought 12 yards of cloth for $15. How many yards must he buy at $$1\frac{3}{4}$, and $$\frac{3}{4}$ a yard, that the average price of the whole may be $$1\frac{1}{5}$ a yard?

4. How much water will dilute 9 gal. 2 qt. 1 pt. of alcohol 96 % strong to 84 %?

5. A grocer mixed teas worth $.30, $.55, and $.70 per pound, forming a mixture worth $.45 per pound, having equal parts of the first two kinds, and 12 pounds of the third kind. How many pounds of each of the first two kinds did he take?

6. How many gallons of water must I mix with 36 gallons of vinegar worth $.35 per gallon to make a mixture worth $.30 per gallon?

7. How many pounds of tea worth $.45 and $.50 a lb. respectively, must be mixed with 20 lb. at $.30 a lb. to make a mixture worth $.40 a lb.?

506. To find the proportion when the quantities of two or more of the ingredients are limited.

1. How many bushels of wheat at \$.75 a bushel and of peas at \$2.00 a bushel, must be mixed with 18 bushels of oats at \$.40, 8 bushels of corn at \$.50, and 4 bushels of barley at \$.65, that the mixture may be worth \$.76 per bushel?

OPERATION.

$$\begin{array}{rcr} \$.40 \times 18 = & \$ & 7.20 \\ .50 \times \ \ 8 = & & 4.00 \\ .65 \times \ \ 4 = & & 2.60 \\ \hline 30\ ) & \$ & 13.80 \\ \hline \text{Mean price of the given simples} & \$ & .46 \end{array}$$

		1	2	3	4	5	6
	46	$\frac{1}{30}$		62		62	$62 \times \frac{30}{62} = 30$
76	75		1		124	124	$124 \times \frac{30}{62} = 60$
	200	$\frac{1}{124}$	$\frac{1}{124}$	15	1	16	$16 \times \frac{30}{62} = 7\frac{23}{31}$

SOLUTION. — Of the given quantities there are $18 + 8 + 4 = 30$ bushels, whose mean or average price we find by § 187 to be \$.46. We are therefore required to mix 30 bushels of grain worth \$.46 per bushel, with wheat at \$.75 and peas at \$2.00 to make a mixture worth \$.76 a bushel. Proceeding as in § 505, we find there will be required 60 bushels of wheat and $7\frac{23}{31}$ bushels of peas.

RULE. — *Consider those ingredients whose quantities and values are given as forming a mixture, and find their mean value by* § 187; *then consider this mixture as a single ingredient whose quantity and value are known, and find the quantities of the other ingredients by* § 505.

If in reducing the fractions to integers, we do not multiply by their least common denominator, different mixtures may result which will also answer the conditions. This will always be the case except when each couplet is multiplied by the same multiple of its least common denominator. Thus in the example given above, if we made column 3, 124 (2×62) and 30 (2×15) and column 4, 248 (2×124) and 2 (2×1) the answer would be the same. The following are illustrations of different results.

[redacted]

OPERATION.

		1	2	3	4	5	6
	46	$\frac{1}{30}$		124		124	$124 \times \frac{30}{124} = 30$
76	75		1		124	124	$124 \times \frac{30}{124} = 30$
	200	$\frac{1}{124}$	$\frac{1}{124}$	30	1	31	$31 \times \frac{30}{124} = 7\frac{1}{2}$
	46	$\frac{1}{30}$		62		62	$62 \times \frac{30}{62} = 30$
76	75		1		1860	1860	$1860 \times \frac{30}{62} = 900$
	200	$\frac{1}{124}$	$\frac{1}{124}$	15	15	30	$30 \times \frac{30}{62} = 14\frac{16}{31}$

SOLUTION.—In the first case, first couplet $\frac{1}{\cancel{30}} \times \cancel{30} \times 124 = 124$ and $\frac{1}{\cancel{124}} \times 30 \times \cancel{124} = 30$. In the second couplet we multiply 1 and $\frac{1}{124}$ each by 124. Adding columns 3 and 4 and then proceeding as before we have the proportion 30, 30, and $7\frac{1}{2}$. These results prove to be correct, for $(30 \times \$.46) + (30 \times \$.75) + (7\frac{1}{2} \times \$2.00) = \$51.30$; $30 + 30 + 7\frac{1}{2} = 67\frac{1}{2}$; $\$51.30 \div 67\frac{1}{2} = \$.76$, the required price of a bushel of the mixture.

In the second case we multiply the first couplet by its least common denominator, 1860, and obtain 62 and 15; we also multiply the second couplet by 1860 and obtain 1860 and 15. Proceeding as before we find the proportion to be 30, 900, $14\frac{16}{31}$. These results prove to be correct, for $(30 \times \$.46) + (900 \times \$.75) + (14\frac{16}{31} \times \$2.00) = \$717.83\frac{7}{31}$; $30 + 900 + 14\frac{16}{31} = 944\frac{16}{31}$. $\$717.83\frac{7}{31} \div 944\frac{16}{31} = \$.76$. Hence it is evident that an indefinite number of correct results might be obtained.

2. A man bought 7 yards of cloth @ $ 2.20, and 7 yards @ $ 2. How much must he buy @ $ 1.60, and @ $ 1.75 that the average price of the whole may be $ 1.80?

3. How much sugar at $ .07 per pound and $ .03 per pound must be mixed with 5 lb. @ $ .04 and 2 lb. @ $ .08 to make a mixture worth $ .06 per pound?

4. A farmer has 40 bushels of wheat worth $ 2.00 a bushel, and 70 bushels of corn worth $ $\frac{1}{2}$ a bushel. How many oats worth $ $\frac{1}{4}$ a bushel must he mix with the wheat and corn to make the mixture worth $ .60 a bushel?

5. A man bought 2 pounds of tea at $ .40 a pound and 6 pounds at $ .60 a pound. How much must he buy at $ .75 a pound and $ .69 a pound respectively to make a mixture worth $ .65 a pound?

507. To find the proportion when the quantity of the whole compound is limited.

1. A tradesman has three kinds of tea, worth \$.30, \$.45, and \$.60 per pound, respectively. What quantities of each should he take to form a mixture of 72 pounds, worth \$.40 per pound?

OPERATION.

		1	2	3	4	5	6
40	30	$\frac{1}{10}$	$\frac{1}{10}$	2	1	3	36
	45		$\frac{1}{5}$		2	2	24
	60	$\frac{1}{20}$		1		1	12
						6	72

SOLUTION. — By § 503 we find the proportional quantities to form the mixture to be 3 lb. at \$.30, 2 lb. at \$.45, and 1 lb. at \$.60. Adding these proportional quantities, we find that they would form a mixture of 6 pounds. And since the required mixture is $\frac{72}{6}$ or 12 times 6 pounds, we multiply each of the proportional terms by 12, and obtain for the required quantities, 36 lb. at \$.30, 24 lb. at \$.45, and 12 lb. at \$.60.

RULE. — *Find the proportional numbers as in § 503. Divide the given quantity by the sum of the proportional quantities and multiply each of the proportional quantities by the quotient thus obtained.*

NOTE. — When the sum of the proportional parts is not an exact divisor of the given quantity, each couplet must be multiplied by such numbers as will make the sum of the proportional parts an exact divisor of the whole quantity.

2. A grocer has coffee worth \$.20, \$.25, and \$.40 per pound, respectively. How much of each kind must he use to fill a cask holding 250 lb. that shall be worth \$.30 a pound?

3. A man bought 154 calves, sheep, and lambs for \$154. He paid \$$3\frac{1}{2}$ for each calf, \$$1\frac{1}{3}$ for each sheep, and \$$\frac{1}{2}$ for each lamb. How many did he buy of each kind?

4. A man paid \$1.65 to 55 laborers, consisting of men, women, and boys; to the men he paid \$5 a week, to the women \$1 a week, and to the boys \$$\frac{1}{2}$ a week. How many were there of each?

5. A man who sells turkeys at \$.75 each, geese at \$.50, ducks at \$.35, and chickens at \$.20, receives \$56 for 140 fowls. How many of each does he sell?

SUGGESTION. — $\frac{1}{35} : \frac{1}{5} :: \frac{1}{2} : 3\frac{1}{2}$. If we make the couplets $\frac{1}{2}$, $3\frac{1}{2}$, 2, and 1 *their sum will be* 7, which is an exact divisor of 140.

PERCENTAGE.

508. **Per Cent** is a contraction of the Latin phrase *per centum*, and signifies *by the hundred;* that is, a certain part of every hundred of any denomination. Thus, 4 per cent means 4 of every hundred.

509. **Percentage** embraces all processes of computation in which the basis of comparison is *one hundred.*

510. The character, %, is generally employed in business transactions to represent the words *per cent;* thus 6% signifies 6 per cent.

511. Since any per cent is some number of hundredths, it is properly expressed by a *decimal fraction;* thus 5 per cent = 5% = .05. Per cent may also be expressed, however, by a *common fraction,* as follows:

		Decimals.		Common Fractions.		Lowest Terms.
1 %	=	.01	=	$\frac{1}{100}$	=	$\frac{1}{100}$
20 %	=	.20	=	$\frac{20}{100}$	=	$\frac{1}{5}$
$12\frac{1}{2}$ %	=	.125	=	$\frac{125}{1000}$	=	$\frac{1}{8}$
100 %	=	1.00	=	$\frac{100}{100}$	=	1
$\frac{3}{4}$ %	=	.0075	=	$\frac{75}{10000}$	=	$\frac{3}{400}$

Examples.

512. Express decimally and by common fractions in their lowest terms:

1. 3%.
2. 9%.
3. 16%.
4. 75%.
5. 15%.
6. 37%.
7. 125%.
8. 184%.
9. $4\frac{1}{2}$%.
10. $5\frac{1}{4}$%.
11. $8\frac{3}{4}$%.
12. $24\frac{1}{3}$%.
13. $\frac{1}{4}$%.
14. $\frac{3}{4}$%.
15. $\frac{1}{2}$%.
16. $\frac{2}{5}$%.
17. $11\frac{1}{9}$%.
18. $37\frac{1}{2}$%.
19. $42\frac{6}{7}$%.
20. $87\frac{1}{2}$%.

21. What per cent is .0725?

Solution. — $.0725 = .07\frac{1}{4} = 7\frac{1}{4}\%$ *Ans.*

What per cent is:

22. .065?	25. .014?	28. .028?	31. $.004\frac{6}{11}$?
23. .14375?	26. .1025?	29. .1324?	32. $.003\frac{1}{12}$?
24. .0975?	27. .004?	30. $.084\frac{2}{7}$?	33. $.05\frac{1}{6}$?

Change the following fractions to equivalent per cents:

34. $\frac{3}{5}$.	36. $\frac{1}{8}$.	38. $\frac{2}{3}$.	40. $\frac{1}{9}$.	42. $\frac{1}{12}$.
35. $\frac{1}{4}$.	37. $\frac{1}{7}$.	39. $\frac{5}{6}$.	41. $\frac{1}{20}$.	43. $\frac{3}{7}$.

513. In the operations of Percentage there are five parts or elements, namely: Rate Per Cent, Percentage, Base, Amount, and Difference.

514. **Rate Per Cent**, or **Rate**, is the decimal which denotes how many hundredths of a number are to be taken.

Notes. — 1. Such expressions as 6 per cent, and 5%, are essentially *decimals*, the words *per cent*, or the character %, indicating the decimal denominator.

2. — If the decimal is reduced to a common fraction in its *lowest terms*, this fraction will still be the equivalent *rate*, though not the rate *per cent*.

515. The **Percentage** is that part of any number which is indicated by the per cent.

516. The **Base** is the number on which the percentage is computed.

517. The **Amount** is the sum obtained by adding the percentage to the base.

518. The **Difference** is the remainder obtained by subtracting the percentage from the base.

519. Corresponding to the five elements in percentage, there are five **Problems**, which may be stated as follows:

1. *Given the base and rate, to find the percentage, amount, and difference.*
2. *Given the percentage and base, to find the rate.*
3. *Given the percentage and rate, to find the base.*
4. *Given the amount and rate, to find the base.*
5. *Given the difference and rate, to find the base.*

[redacted]

Examples.

520. Given the base and rate, to find the percentage, amount, and difference.

1. What is 4% of $125, and what are the amount and difference?

OPERATION.

$$\begin{array}{r} \$125 \\ .04 \\ \hline \$5.00 \end{array}$$, Percentage. Or, $\frac{1}{25}$ of $125 = $5, Percentage.

$125 + $5 = $130, Amt. $125 − $5 = $120, Diff.

Or,

100% + 4% = 104%; 1.04 of $125 = $130, Amt.

100% − 4% = 96%; .96 of $125 = $120, Diff.

SOLUTION.—4% of $125 = .04 of $125, or $5. Or 4% = $\frac{1}{25}$; $\frac{1}{25}$ of $125 = $5, percentage.

Since the amount is the base plus the percentage, it is $125 + $5 = $130; and since the difference is the base minus the percentage, it is $125 − $5 = $120. Or, the base 100% + the rate 4% = 104%, the amount expressed in per cent. 104% of $125 = 1.04 of $125 = $130, Amt.; and the base 100% − the rate 4% = 96%, the difference expressed in per cent. 96% of $125 = .96 of $125 = $120, Diff.

RULE.—I. To find the percentage.—*Multiply the base by the rate per cent.*

II. To find the amount.—*Add the percentage and base.*

III. To find the difference.—*Subtract the percentage from the base.*

IV. To find the amount or difference directly from the base.—*Add the rate per cent to 100%, and multiply the base by this per cent, expressed decimally, to find the amount; and subtract the rate from 100%, and multiply the base by the result to find the difference.*

NOTES.—1. When the given per cent is an aliquot part of 100, multiply by the aliquot part expressed by a fraction in its lowest term. Thus, instead of multiplying by $33\frac{1}{3}$% take $\frac{1}{3}$ of the number. If the per cent is $87\frac{1}{2}$, take $\frac{7}{8}$ of the number, etc.

2. To find 10% (= $\frac{1}{10}$) of a number, move the decimal point one place to the left.

What is:

2. 4% of 250?	5. $12\frac{1}{2}$% of $5600?	8. 75% of 487 bu.?
3. 7% of 350?	6. 9% of 785 lb.?	9. $33\frac{1}{3}$% of 275 men?
4. 16% of 324?	7. 25% of 960 mi.?	10. $\frac{1}{4}$% of $2364?

Find:

11. 105% of $5760.

12. $3\frac{2}{3}$% of $856.

13. $112\frac{1}{2}$% of 2450.

14. $66\frac{2}{3}$% of 846.

15. A man owes $536 to A, $450 to B, and $784 to C. How much money will be required to pay 54% of his debts?

16. My salary is $1500 a year. If I pay 15% for board, 5% for clothing, 6% for books, and 8% for incidentals, what are my yearly expenses and how much have I left?

NOTE. — 15% + 5% + 6% + 8% = 34%. In all cases where several rates refer to the same *base*, they may be added or subtracted, according to the conditions of the question.

17. A man having a yearly income of $3500, spends 10% of it the first year, 12% the second year, and 18% the third year. How much does he save in the 3 years?

18. A had $6000 in a bank. He drew out 25% of it, then 30% of the remainder, and afterward deposited 10% of what he had drawn. How much had he then in bank?

19. A began business, Jan. 1, with a capital of $5400, and at the end of 1 year his ledger showed the condition of his business as follows: For Jan., 2% gain; Feb., $3\frac{1}{4}$% gain; March, $\frac{1}{2}$% loss; Apr., 2% gain; May, $2\frac{1}{2}$% gain; June, $1\frac{3}{4}$% loss; July, $1\frac{1}{2}$% gain; Aug., 1% loss; Sept., $2\frac{3}{4}$% gain; Oct., 4% gain; Nov., $\frac{3}{4}$% loss; Dec., 3% gain. Find the profits and the amount of the capital at the end of the year.

521. Given the percentage and base, to find the rate.

1. What per cent of 360 is 18?

OPERATION.

$18 \div 360 = .05 = 5\%$ *Ans.*

Or,

$\frac{18}{360} = \frac{1}{20} = .05 = 5\%$ *Ans.*

SOLUTION. — Since the percentage is always the *product* of the base and rate (§ 520), we divide the given percentage, 18, by the given base, 360, and obtain the required rate, .05 = 5%.

RULE. — *Divide the percentage by the base.*

What per cent of:

2. $720 is $21.60?

3. 1560 lb. is 234 lb.?

4. 980 rd. is 49 rd.?

5. 46 gal. is 5 gal. 3 qt.?

6. 7.85 mi. is 5.495 mi.?

7. $\frac{5}{16}$ is $\frac{3}{8}$?

8. An editor having 5600 subscribers, lost 448. What was his loss per cent?

9. A merchant owes \$7560, and his assets are \$4914. What per cent of his debts can he pay?

10. A man shipped 2600 bushels of grain from Chicago, and 455 bushels were thrown overboard during a gale. What was the rate per cent of his loss?

11. A miller having 720 barrels of flour, sold 288 barrels. What per cent of his stock remained unsold?

12. In a class examination, 165 questions were submitted to each of the 5 members; A answered 130, B 125, C 96, D 110, and E 160. What was the standing of the class?

522. Given the percentage and rate, to find the base.

1. 18 is 5 % of what number?

OPERATION.

$18 \div .05 = 360$ *Ans.*

Or,

$18 \div \frac{1}{20} = 360$ *Ans.*

SOLUTION. — Since the percentage is always the *product* of the base and rate (§ 520), we divide the given percentage, 18, by the given rate, .05, or $\frac{1}{20}$, and obtain the base, 360.

RULE. — *Divide the percentage by the rate.*

What is the number of which:

2. 18 is 25 %?
3. 54 is 15 %?
4. 17.5 is $2\frac{1}{3}$%?
5. 2.28 is 5 %?
6. 414 is 120 %?
7. 6119 is $105\frac{1}{2}$%?

8. The percentage is \$18.75, and the rate is $2\frac{1}{2}$%. What is the base?

9. The percentage is $31\frac{1}{4}$, and the rate $31\frac{1}{4}$%. What is the base?

10. I sold my house for \$4578, which was 84% of its cost. What was the cost?

11. A wool grower sold 3150 head of sheep, and had 30% of his original stock left. How many sheep had he at first?

12. A man drew 40% of his bank deposits, and expended $13\frac{1}{3}$% of the money thus drawn in the purchase of a carriage worth \$116. How much money had he in bank?

13. If \$147.56 is $13\frac{1}{8}$% of A's money, and $4\frac{3}{8}$% of A's money is 8% of B's, how much more money has A than B?

14. In a battle 4% of the army were slain upon the field; and 5% of the remainder died of wounds, in the hospital. The difference between the killed and the mortally wounded was 168. How many men were there in the army?

SUGGESTION. — 100% − 4% = 96%, left after the battle; and 5% of 96% = $4\frac{4}{5}$%, the part of the army that died of wounds. The difference between $4\frac{4}{5}$% and 4% is $\frac{4}{5}$%.

15. A owns $\frac{3}{4}$ of a prize and B the remainder; after A has taken 40% of his share, and B 20% of his share, the remainder is equitably divided between them by giving A $1950 more than B. What is the value of the prize?

523. Given the amount and rate, to find the base.

1. What number increased by 5% of itself is equal to 378?

OPERATION.

$1 + .05 = 1.05$

$378 \div 1.05 = 360$ *Ans.*

Or,

$1\frac{1}{20} = \frac{21}{20}$

$378 \div \frac{21}{20} = 360$ *Ans.*

SOLUTION. — If any number is increased by 5% of itself, the amount will be 1.05 times the number. We therefore divide the given *amount*, 378, by 1.05, or $\frac{21}{20}$, and obtain the *base*, 360, which is the number required.

RULE. — *Divide the amount by* 1 *plus the rate.*

NOTE. — The amount is always a *product*, of which the base is one factor, and 1 plus the rate the other factor.

2. What number added to 15% of itself is equal to 644?

3. A has $815.36, which is 4% more than B has. How much money has B?

4. Having increased my stock in trade by 12% of itself, I find that I have $3800. How much had I at first?

5. In 1890 the population of Albany was 164555, which was an increase of $6\frac{6}{25}$% over that of 1880. What was the population in 1880?

6. My crop of wheat this year is 8% greater than my crop of last year, and I have raised during the two years 5200 bushels. What was my last year's crop?

SUGGESTION. — 1.00 + 1.08 = 2.08. Hence, 5200 bu. = 2.08 of last year's crop.

7. The net profits of a nursery in two years were $6970, and the profits the second year were 5% greater than the profits the first year. What were the profits each year?

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8. If a number is increased 8%, and the amount is increased 7%, the result will be 86.67. What is the number?

SUGGESTION. — The whole amount will be $1.08 \times 1.07 = 1.1556$ times the original number.

9. A produce dealer bought grain by measure, and sold it by weight, thereby gaining $1\frac{1}{2}$% in the number of bushels. He sold at a price 5% above his buying price, and received $ 4910.976 for the grain. What was the cost?

10. B has 6%, and C 4% more money than A, and they all have $ 11160. How much money has A?

11. In the erection of a house I paid twice as much for material as for labor. Had I paid 6% more for material, and 9% more for labor, my house would have cost $ 1284. What was its cost?

524. Given the difference and rate, to find the base.

1. What number diminished by 5% of itself, is equal to 342?

OPERATION.

$1 - .05 = .95$

$342 \div .95 = 360$ *Ans.*

Or,

$1 - \frac{1}{20} = \frac{19}{20}$

$342 \div \frac{19}{20} = 360$ *Ans.*

SOLUTION. — If any number is diminished by 5% of itself, the difference will be .95 of the number. We therefore divide the given difference, 342, by .95, or $\frac{19}{20}$, and obtain the base, 360, which is the required number.

RULE. — *Divide the difference by* 1 *minus the rate.*

NOTE. — The difference is always a *product*, of which the base is one factor, and 1 minus the rate the other.

2. What number less 10% of itself equals 504?

3. The rate is 8% and the difference $ 4.37. What is the base?

4. After taking away 15% of a heap of grain, there remained 40 bu. $3\frac{1}{8}$ pk. How many bushels were there at first?

5. Having sold 36% of my land, I have 224 acres left. How much land had I at first?

6. After paying 65% of my debts, I find that $ 2590 will discharge the remainder. How much did I owe in all?

7. A young man having received a fortune, deposited 80% of it in a bank. He afterward drew 20% of his deposit, and then had $ 5760 in bank. What was his entire fortune?

8. A man owning $\frac{7}{8}$ of a ship, sold 12% of his share to A, and the remainder to B, at the same rate, for $ 20020. What was the estimated value of the whole ship?

9. An army which has been twice decimated in battle, now contains only 6480 men. What was the original number?

10. Each of two men, A and B, desired to sell his horse to C. A asked a certain price, and B asked 50% more. A then reduced his price 20%, and B his price 30%, at which prices C took both horses, paying for them $ 148. What was each man's asking price?

11. A buyer expended equal sums of money in the purchase of wheat, corn, and oats. In the sales, he cleared 6% on the wheat, and 3% on the corn, but lost 17% on the oats; the whole amount received was $ 2336. What sum did he lay out in each kind of grain?

525. If we use the initial letters for base, percentage, rate, amount, and difference, the cases in percentage may be briefly stated by the following formulas:

$$p = b \times r.$$
$$a = b + p \text{ or } b \times (1 + r).$$
$$d = b - p \text{ or } b \times (1 - r).$$
$$r = p \div b$$
$$b = p \div r = a \div (1 + r) = d \div (1 - r).$$

526. The chief applications of Percentage, where time is not considered, are Profit and Loss, Commission and Brokerage, Trade Discount, Insurance, Taxes, General Average, Duties and Customs, Stocks and Bonds. Since the five problems in Percentage involve all the essential relations of the parts or elements, we have for the above applications the following rule:

GENERAL RULE. — *Note what elements of Percentage are given in the example, and what element is required; then apply the special rule for the corresponding case.*

PROFIT AND LOSS.

527. **Profit** and **Loss** are commercial terms, used to express the gain or loss in business transactions.

528. Gains and losses are usually estimated at some rate per cent on the money first expended or invested.

The calculations in profit and loss are based on the following relations:

I. The *Cost* corresponds to the *Base.*

II. The *Per Cent* of profit or loss corresponds to the *Rate Per Cent.*

III. The *Profit* or *Loss* corresponds to the *Percentage.*

IV. The *Selling Price* corresponds to the *Amount* or *Difference,* according as it is greater or less than the cost.

NOTE.—The *prime cost* is the actual cost of the article bought; the *gross cost* or *entire cost* is the total cost including expenses incurred for freight, commission, repairs, etc.

Examples.

529. 1. A merchant bought cloth for \$ 3.25 per yard, and gained 8% in selling. Find the gain and the selling price. If he lost 8%, find the loss and selling price.

OPERATION.

(1)

$\$3.25 \times .08 = \$.26$, Gain.

$\$3.25 + \$.26 = \$3.51$, Selling Price.

Or,

$\$3.25 \times 1.08 = \$3.51$, Selling Price.

(2)

$\$3.25 \times .08 = \$.26$, Loss.

$\$3.25 - \$.26 = \$2.99$, Selling Price

Or,

$\$3.25 \times .92 = \$2.99$, Selling Price.

SOLUTION.—According to § 520, we multiply \$3.25, the cost or base, by .08, and we have \$.26 for the gain or loss. This, added to the cost, gives the selling price of the first example, and, subtracted from the cost, the selling price of the second.

Or, we may find the selling price directly from the cost by multiplying the cost by 1 plus the rate where there is a gain, and by 1 minus the rate where there is a loss (§ 520, IV.).

2. A jobber invested \$2560 in dry goods, and realized \$384 net profit. What was the rate per cent of his gain? If he had lost \$384, what would have been the rate of his loss?

OPERATION.

\$384 ÷ \$2560 = 15% *Ans.*

SOLUTION. — According to § 521, we divide the gain or loss, \$384, which is the *percentage*, by the cost, \$2560, which is the *base*, and obtain .15 = 15%, the *rate* of gain or loss.

3. A jeweler gained (or lost) \$25 on a necklace, which was 5% of the cost. What was the cost?

OPERATION.

\$25 ÷ .05 = \$500 *Ans.*

SOLUTION. — According to § 522, we divide the gain or loss, which is the *percentage*, by the rate, and obtain \$500, the *base* or cost.

4. A dealer sold wheat at a gain of 5% and realized \$9030. What was the cost?

OPERATION.

1.00 + .05 = 1.05

\$9030 ÷ 1.05 = \$8600 *Ans.*

SOLUTION. — According to § 523, we divide the selling price or *amount*, \$9030, by 1 plus the rate of gain, or 1.05, and obtain the *base* or cost, \$8600.

5. A produce dealer sold a shipment of wheat at a loss of 5%, realizing as the net proceeds, \$8170. What was the cost?

OPERATION.

1.00 − .05 = .95

\$8170 ÷ .95 = \$8600 *Ans.*

SOLUTION. — According to § 524, we divide the net proceeds, \$8170, which is the *difference*, by 1 minus the *rate* of loss, or .95, and obtain the *base* or cost, \$8600.

6. A merchant pays \$7650 for a stock of spring goods. If he sells at an advance of 20% upon the purchase price, what will be his profits, after deducting \$480 for expenses?

7. A merchant bought 320 yards of calico at \$.15, and sold it at a reduction of $2\frac{1}{2}$%. What was the entire loss?

8. A quantity of corn was bought for \$.50 a bushel. At what price must it be sold to gain $33\frac{1}{3}$%?

9. Some fish bought for \$4.25, were sold for \$4.93. What was the gain per cent?

10. A sloop, freighted with 3840 bu. of corn, threw $37\frac{1}{2}$% of her cargo overboard. What was the loss, at \$.$62\frac{1}{2}$ a bushel?

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11. A man bought a pair of horses for \$ 275, and sold them for \$ 330. What per cent did he gain?

12. If a merchant buys cloth at \$ .60 a yard, and sells it for \$ .75 a yard, what per cent does he gain?

13. A, having a debt against B, agreed to take \$ .87$\frac{1}{2}$ on the dollar. What per cent did A lose?

14. A quantity of flour was sold for \$ 1881, which was 18$\frac{3}{4}$% more than it cost. How much did it cost?

15. I sold 25 barrels of apples for \$ 69.75, and made 24%. How much did they cost me per barrel?

16. A grocer sold 4 barrels of sugar for \$ 24 each; on 2 barrels he gained 20%, and on the other 2 he lost 20%. Did he gain or lose on the whole, and how much?

17. I sold a farm of 106 A. 30 sq. rd. for \$ 96 an acre, and gained 18% on the cost. How much did the whole farm cost?

18. A lumberman sold 36840 feet of lumber at \$ 21.12 per M, and gained 28%. How much would he have gained or lost, had he sold it at \$ 17 per M?

19. I sold my carriage at 30% gain, and with the money bought another, which I sold for \$ 182, and lost 12$\frac{1}{2}$%. How much did each carriage cost me?

20. Gaffney, Burke & Co. bought a quantity of dry goods for \$ 6840; they sold $\frac{1}{4}$ of them at 15% profit, $\frac{1}{3}$ at 18$\frac{3}{4}$%, $\frac{1}{6}$ at 20%, and the remainder at 33$\frac{1}{3}$% profit. How much was the average gain per cent, and how much the whole gain?

21. If I buy a piece of land, and it increases in value each year at the rate of 50% on the value of each previous year, for 4 years, and at the end of this time is worth \$ 12000, how much did it cost?

22. A man buys some goods for 20% below cost, and sells them for 16% below cost. What is his rate of gain?

23. A machinist sold 24 grain drills for \$ 125 each. On one half of them he gained 25%, and on the remainder he lost 25%. Did he gain or lose on the whole, and how much?

24. A Western merchant bought a quantity of Red Southern wheat for \$1080, White Michigan for \$1950, and Chicago Spring for \$250. He shipped the whole to his correspondent in Buffalo, who sold the first two kinds at an advance of 20% in the price, and the balance at a loss of \$10, then deducting from the gross avails his commission of 5%, and \$254.60 for expenses, he returned to the consignor the net proceeds. What was the rate of the merchant's gain?

25. I bought land at \$30 an acre. How much per acre must I ask for it, that I may abate 25% from my asking price, and still make 20 per cent on the purchase money?

26. A salesman asked an advance of 20% on the cost of some goods, but was obliged to sell at 20% less than his asking price. Did he gain or lose, and what was his per cent of gain or loss?

27. A grocer sold a hogshead of molasses for \$31.50, which was a reduction of 30% from the prime cost. What was the purchase price paid per gallon?

28. A drygoods merchant sells calicos for \$.02½ per yard more than they cost, and realizes a profit of 8%. What is the cost per yard?

29. If I make a profit of 18¾% by selling broadcloth for \$.75 per yard above cost, how much must I advance on this price to realize a profit of 31¼%?

30. A speculator gained 30% on $\frac{3}{5}$ of his investment, and lost 5% on the remainder, and his net profits were \$720. What would have been his profits, had he gained 30% on $\frac{2}{5}$ and lost 5% on the remainder?

31. A man wishing to sell his real estate asked 36% more than it cost him, but he finally sold it for 16% less than his asking price. He gained by the transaction \$740.48. How much did the real estate cost him, what was his asking price, and for how much did he sell it?

32. I sold $\frac{5}{8}$ of a barrel of beef for what the whole barrel cost. What per cent did I gain on the part sold?

COMMISSION AND BROKERAGE.

530. **Commission** is the *percentage* of fee or compensation paid an agent for services rendered.

531. **Brokerage** is the fee or allowance paid to a broker, or dealer in money, stocks, or bills of exchange, for making exchanges of money, buying and selling stocks, negotiating bills of exchange, or transacting other like business.

532. An **Agent, Commission Merchant,** or **Broker,** is a person who transacts business for another, or buys and sells goods, money, stocks, notes, etc.

533. A **Consignment** is a quantity of goods sent to one person to be sold on commission for another person.

534. A **Consignee** is a person who receives goods to sell for another; and a **Consignor** is a person who sends goods to another to be sold.

535. The **Net Proceeds** of a sale or collection is the sum left, after deducting the commission and other charges.

Note. — A person who is employed in establishing mercantile relations between others living at a distance from each other, is called the *correspondent* of the party in whose behalf he acts. A correspondent is the *agent* of those whose custom or patronage he secures to the party in whose interest he is employed.

536. The rates of commission and brokerage are not regulated by law, but are usually reckoned at a certain per cent upon the money employed in the transaction. When an agent *sells,* his commission is some per cent of the *sales;* and when he *buys,* it is some per cent of the *purchase price.* The calculations in commission and brokerage are based on the following relations:

I. *Commission* is *Percentage.*

II. The *Sum received* by the agent as the price of property sold, or the *Sum invested* by the agent in the purchase or exchange of property, is the *Base* of commission.

III. The *Sum remitted* to an agent, and including both the purchase money and the agent's commission, is the *Amount.*

IV. The sum due the employer or consignor as the *Net Proceeds* of a sale or collection, is the *Difference.*

Examples.

537. 1. My agent sells goods to the amount of $6250. What is his commission at 3%?

OPERATION.

$\$6250 \times .03 = \$187.50$ *Ans.*

SOLUTION.—According to § 520, we multiply the sum obtained for the goods, $6250, which is the *base* of the commission by the rate of the commission, .03, and obtain the *commission* or *percentage*, $187.50.

2. A flour merchant remits to his agent in Chicago, $3796, for the purchase of grain, after deducting his commission of 4%. How much will the agent expend for his employer, and what will be his commission?

OPERATION.

$$1.00 + .04 = 1.04$$

$$\left.\begin{array}{l}\$3796 \div 1.04 = \$3650\text{, for grain,} \\ \$3796 - \$3650 = \$146\text{, Com.}\end{array}\right\} \textit{Ans.}$$

SOLUTION. — According to § 523, we divide the remittance, $3796, which is the *amount*, by 1 plus the *rate* of commission, or 1.04, and obtain the *base* of commission, $3650, which is the sum to be expended in the purchase. Subtracting this from the remittance, we have $146, the commission.

NOTE.—It is evident that the whole remittance, $3796, should not be taken as the base of commission: for that would be computing commission on commission. A person must charge commission only on what he *expends* or *collects*, in his capacity as agent.

3. An agent sold real estate on commission of 5%, and returned to the owner, as the net proceeds, $8075. For what price did he sell the property, and what was his commission?

OPERATION.

$$1.00 - .05 = .95$$

$$\left.\begin{array}{l}\$8075 \div .95 = \$8500\text{, Price,} \\ \$8500 - \$8075 = \$425\text{, Com.}\end{array}\right\} \textit{Ans.}$$

SOLUTION.—According to § 524, we divide the net proceeds, $8075, which is the *difference* by 1 minus the *rate* of commission, and obtain the *base*, $8500, which is the price of the property sold; whence by subtraction, we obtain the commission, $425.

4. An agent sold my house and lot for $8600. What was his commission at $2\frac{1}{4}$%?

5. A lawyer collects $750.75. What is his commission at $3\frac{3}{4}$%?

6. My agent in New York has sold a quantity of Indiana wheat for $ 4900, and of corn for $ 2664. What is his commission at $2\frac{1}{4}\%$?

7. A commission merchant sold a consignment of flour and pork for $ 25372. He charged $ 132 for storage, and $6\frac{1}{4}\%$ commission. What were the net proceeds of the sale ?

8. An agent for a Rochester nurseryman sells 4000 apple trees at $ 25 per hundred, 2000 pear trees at $ 50 per hundred, 1600 peach trees at $ 20 per hundred, 1800 cherry trees at $ 50 per hundred, and 500 plum trees at $ 50 per hundred. What is his commission at 30%, and how much should he return to his employer as the net proceeds, after deducting $ 203.50 for expenses ?

9. A lawyer having a debt of $ 785 to collect, compromises for 82%. What is his commission at 5% ?

10. An agent received $ 63 for collecting a debt of $ 1260. What was the rate of his commission ?

11. My Charleston agent charged $ 74.25 for purchasing 26400 lb. of rice at $ 4.50 per 100 lb. What was the rate of his commission ?

12. A house and lot were sold for $ 7850, and the owner received $ 7732.25 as the net proceeds. What was the rate of commission ?

13. A commission merchant in Boston sells cotton to the amount of $ 3500. After deducting $ 35.36 for freight and cartage, $ 10.50 for storage, and his commission, he remits to his employer $ 3252.89 as the net proceeds of the sale. At what rate did he charge commission ?

14. The net proceeds of a sale were $ 5635, and the commission was $ 115. What was the rate of commission ?

15. An agent received $ 22.40 for selling grain at a commission of 4%. What was the value of the grain sold ?

16. My attorney, in collecting a note for me at a commission of 8%, received as his fee $ 6.80. What was the value of the note ?

17. A real estate agent sold a house for a certain sum of money and remitted $ 19600 to the owner, after deducting his commission of 2%. For how much did the agent sell the house?

18. An agent sold a piano. After deducting his commission of 10% he sent the owner $ 450. How much did the purchaser pay for it?

19. I sent to my agent in Boston $ 255, to be invested in calico at $ .15 per yard, after deducting his commission of 2%. How many yards shall I receive?

20. John Kennedy, commission merchant, sells for Ladd & Co. 860 barrels of flour @ $ 3.50, on a commission of $2\frac{1}{2}$%. He invests the proceeds in dry goods, after deducting his commission of $1\frac{1}{2}$% for purchasing. How many dollars' worth of goods do Ladd & Co. receive?

21. A commission merchant, whose rate both for selling and investing is 5%, receives 24000 lb. of pork, worth $ .06 a lb., and $ 3000 in cash, with instructions to invest in a shipment of cotton to London. What will be his entire commission?

22. A speculator received $ 3290 as the net proceeds of a sale, after allowing a commission of 6%. What was the value of the property?

23. The net proceeds of a shipment of hay, after deducting a commission of 3%, and $ 500 for other charges, was $ 6290. What was the selling price?

24. I send a quantity of dry goods into the country to be sold at auction, on commission of 9%. What amount of goods must be sold, that my agent may buy produce with the avails, to the value of $ 3500, after retaining his purchase commission of 4%?

NOTE. — $ 3500 plus the agent's commission equals the net proceeds of the sale.

25. Having sold a consignment of cotton on 3% commission, I am requested to invest the proceeds in city lots after deducting my purchase commission of 2%. My whole commission is $ 265. What is the price of the city lots?

TRADE DISCOUNT.

538. **Trade** or **Commercial Discount** is a deduction from the face of bills, from the list price of goods, or from the amount of a debt, without regard to *time,* and is usually expressed by the term, "*per cent off.*"

Thus, a discount of 25% means a deduction of 25% from the asking price. A discount of 30% and 10% does not mean 40% off, but that 30% is first to be deducted, leaving 70% of the price, then 10% of 70% of the price, equal to 7%, is deducted from the remainder, leaving 63%; so that the total discount is but 37%. 3 *tens* and 5% off means *three* successive discounts of 10% and 5% from the remainder, etc.

539. The **List Price** is called the fixed price, and the discount is sometimes called a **Rebate.**

540. The **Net Price** is the list price, less the discount, or the price *received* for the goods.

NOTES. — 1. Trade discounts are made to avoid the necessity of changing list or catalogue prices when the market price changes; the list price remains the same, and the discounts are changed to meet the rise or fall in prices.

2. *Cash discounts* are discounts made in consideration of immediate payment; *time discounts* are discounts made in consideration of the payment of a bill within a limited time.

541. The calculations of trade discount are based on the following relations:

I. The *List Price* or *Amount of Bill* is the *Base.*
II. The *Rate of Discount* is the *Rate Per Cent.*
III. The *Discount* is the *Percentage.*
IV. The *Net price* is the *Difference.*

Examples.

542. 1. What is the net cost of a bill of goods amounting to \$975, bought at 10% discount, and 5% off for cash?

OPERATION.

$1.00 - .10 = .90$

$.05 \text{ of } .90 = .04\frac{1}{2}$

$.90 - .04\frac{1}{2} = .85\frac{1}{2}$

$\$975 \times .855 = \$833.625$ *Ans.*

SOLUTION. — The first discount of 10% leaves 100% − 10% = 90% of the asking price. The second discount is 5% of 90% or $4\frac{1}{2}$%; and 90% − $4\frac{1}{2}$% = $85\frac{1}{2}$%. Then we have \$975, the *base* or cost, and $85\frac{1}{2}$%, which corresponds to 1 minus the rate, and we find the *net cost* or *difference* by § 520.

Find the *net cost* and *discount* of the following bills:

2. Bought for $750, on 3 mo., at 20% discount, and 4% off for cash.

3. Bought for $365.75, on 4 mo., at 20%, 10%, and 5% off for cash.

4. Bought for $1600 at successive discounts of 15%, 2%, and 5%.

5. Bought for $2340 at successive discounts of $33\frac{1}{3}$%, 5%, and 2%.

6. Bought for $260, on 90 da., at 2 tens and 3% off for cash.

7. What is the difference on a bill of $650, between a discount of 30%, and a discount of 25% and 5% off?

8. What is the price of books which, after a discount of $33\frac{1}{3}$% has been deducted cost $.$66\frac{2}{3}$ each?

9. What is the amount of a bill which, after a rebate of $12\frac{1}{2}$%, amounts to $260.75?

10. What is the net amount of a bill of $169.75 subject to a discount of 30%, 10%, and 5% off for cash?

11. If I buy goods amounting to $2359 at discounts of $33\frac{1}{3}$%, 10%, and 5%, how much do they cost me?

12. I can buy goods from A, B, C, or D at the same list price, but A will allow me successive discounts of 20%, 10%, and 5%; B will allow successive discounts of 5%, 10%, and 20%; C 10%, 20%, and 5%, and D 20%, 5%, and 10%. Which will be the best purchase to make? Explain why.

13. A man buys goods listed at $8.00 at a discount of 20%, and a certain per cent additional for cash. If he pays $6.08 for them, what is the cash discount?

14. A man buys a quantity of goods listed at $250 at a discount of 10% and a certain per cent additional for cash. The net cost of goods is $213.75. What is the cash discount?

15. Which is more profitable, to buy goods costing $2000 at successive discounts of 12%, 5%, and 2%, or 2%, 8%, and 9%, and what is the difference?

INSURANCE.

543. **Insurance** is security guaranteed by one party to another, for a stipulated sum, against damage or risk. It is of two kinds: insurance on property and life insurance.

544. **Property Insurance** includes Fire Insurance, Marine Insurance, and Live Stock Insurance.

Fire Insurance is indemnity for loss of property by fire. **Marine** and **Inland Insurance** are indemnity for the loss of a vessel, or cargo, by casualties of navigation on the ocean or on inland waters. **Stock Insurance** is indemnity for the loss of cattle, horses, etc.

545. The **Insurer** or **Underwriter** is the party taking the risk.

546. The **Insured** or **Assured** is the party protected by the insurance.

547. The **Policy** is the written contract between the parties.

548. The **Premium** is the sum paid by the *insured* to the *insurer*, and in insuring property is estimated at a certain rate per cent of the amount insured, which rate varies according to the degree of hazard, or class of risk.

549. The **Sum Covered** by insurance is the face of the policy.

NOTES 1. — As a security against fraud, most insurance companies take risks at not more than two thirds the full value of the property insured. When insured property suffers damage less than the amount of the policy, the insurers pay only the estimated loss.

2. Insurance business is generally conducted by Joint-stock Companies, or Mutual Companies.

3. A Stock Insurance Company is one in which the capital is owned by individuals called stockholders. They alone share the profits, and are liable for the losses.

4. A Mutual Insurance Company is one in which each person insured is entitled to a share in the profits of the concern, and is liable for the losses. The annual dividends returned by such companies (in case of profits) reduce the premium or cost of insurance; and the assessments made (in case of loss) increase the cost to the policy holder.

550. The calculations in insurance are based upon the following relations:

I. *Premium* is *Percentage*.

II. The *Rate of Premium* is the *Rate Per Cent.*

III. The *Sum Insured* is the *Base* of premium.

IV. The *Sum Covered* by insurance is *Difference*.

Examples.

551. **1.** What premium must be paid for insuring my stock of goods to the amount of \$ 5760 at $1\frac{1}{4}\%$?

OPERATION.

\$ 5760 × .0125 = \$ 72 *Ans.*

SOLUTION.— According to § 520, we multiply \$ 5760, the *base* of premium, by .0125, the rate, and obtain \$ 72, the premium.

2. For what sum must a granary be insured at 2% in order to cover the loss of the wheat, valued at \$ 1617, and the premium?

OPERATION.

$1.00 - .02 = .98$

\$ 1617 ÷ .98 = \$ 1650 *Ans.*

SOLUTION.— According to § 524, we divide the sum to be covered, \$ 1617, which is *difference*, by 1 minus the rate of premium, and obtain \$ 1650, the *base* of premium, or the sum to be insured.

PROOF.— \$ 1650 × .02 = \$ 33, premium; \$ 1650 − \$ 33 = \$ 1617, the sum covered.

3. What must be paid for an insurance of \$ 5860 at $1\frac{1}{2}\%$?

4. What is the premium of \$ 860 at $\frac{1}{2}\%$?

5. What is the premium for an insurance of \$ 3500 on my house and barn, at $1\frac{1}{4}\%$?

6. A fishing craft, insured for \$ 10000 at $2\frac{1}{4}\%$, was totally wrecked. How much of the loss was covered?

7. A hotel valued at \$ 10000 has been insured for \$ 6000 at $1\frac{1}{4}\%$, \$ 5.50 being charged for the policy and the survey of the premises. If it should be destroyed by fire, what loss would the owner suffer?

8. A merchant whose stock in trade is worth \$ 12000, gets the goods insured for $\frac{4}{5}$ of their value, at $\frac{3}{4}\%$. If in a conflagration he saves only \$ 2000 of the stock, what actual loss will he sustain?

9. I pay \$ 12 for an insurance of \$ 800. What is the rate of premium?

10. A trader got a shipment of flour insured for 80% of its cost, at $3\frac{1}{4}\%$, paying \$ 107.25 premium. At what price did he purchase the flour?

11. If I take a risk of \$ 36000 at $2\frac{1}{2}\%$, and reinsure $\frac{1}{2}$ of it at 3%, what is my balance of the premium?

12. What will be the cost of insuring a quantity of wheat valued at \$ 7500, at $\frac{4}{5}\%$?

13. How much will it cost to insure a factory valued at \$ 21000, at $\frac{4}{5}\%$, and the machinery valued at \$ 15400, at $\frac{5}{8}\%$?

14. The Astor Insurance Company took a risk of \$ 16000, for a premium of \$ 280. What was the rate of insurance?

15. A whaling merchant gets his vessel insured for \$ 20000 in the Gallatin Company, at $\frac{3}{4}\%$, and for \$ 30000 in the Howard Company, at $\frac{1}{2}\%$. What rate of premium does he pay on the whole insurance?

16. If it costs \$ 46.75 to insure a store for $\frac{1}{2}$ of its value, at $1\frac{3}{8}\%$, what is the store worth?

17. For what sum must I get my library insured at $1\frac{1}{8}\%$, to cover a loss of \$ 7910?

18. What will be the premium for insuring at $2\frac{3}{7}\%$, property to cover \$ 27320?

19. A shipment of pork was insured at $4\frac{3}{8}\%$, for a sum equal to $\frac{5}{8}$ of its value. The premium paid was \$ 122.50. What was the pork worth?

20. A man obtained an insurance on his house for $\frac{3}{4}$ of its value, at $1\frac{1}{2}\%$ annually. After paying 5 premiums, the house was destroyed by fire, in consequence of which he suffered a loss (including premiums paid) of \$ 2940. What was the value of the house?

21. A man's property is insured at $2\frac{1}{2}\%$ premium, payable annually. In how many years will the premium he has paid equal the policy?

22. A company took a risk at $2\frac{1}{4}\%$, and reinsured $\frac{5}{8}$ of it in another company at $2\frac{1}{2}\%$. The premium received exceeded the premium paid by \$ 72. What was the amount of the risk?

23. The Commercial Insurance Company issued a policy of insurance on an East India merchantman for $\frac{3}{4}$ of the esti-

mated value of the ship and cargo, at 4¼%, and immediately reinsured ½ of the risk in the Manhattan Company at 3%. During the outward voyage the ship was wrecked, and the Manhattan Company lost $1350 more than the Commercial Company. How much did the owners lose?

Life Insurance.

552. **Life Insurance** is a contract by which a company agrees to pay a certain sum of money on the death of an individual, to his heirs (in consideration of an annual premium to be paid during life or for a limited number of years), or to pay the sum to himself, if he survives a certain number of years.

553. The following are the principal kinds of policies issued by a Life Insurance Company:

1. **Life Policies**, payable at the death of the person insured, the annual premium to continue during life, called continued premium life policies.

2. **Life Policies**, payable at the death of the person insured, the annual premium to continue ten years, fifteen years, twenty years, etc., called ten, fifteen, or twenty payment life policies.

3. **Renewable Term Policies**, insuring payment if death occurs within a certain term of years, as ten, fifteen, twenty, etc., which term may be renewed at the expiration of the time by payment of the premium required for the age of the insured.

4. **Endowment Policies**, payable to the person insured, at the end of a certain number of years, as ten, fifteen, twenty, twenty-five, thirty, or thirty-five, or to his heirs, if he dies sooner; annual premium to continue during the existence of the policy.

5. **Endowment Policies**, payable as the preceding, but the payments all to be made in one, five, or ten years.

6. **Annuities**, providing a fixed income continuing only during the lifetime of an annuitant.

[redacted]

Annuities are especially adapted to persons of an advanced age, to women, and to all who wish to avoid the risk of taking care of their funds. Thus, if a person at the age of 50 pays $1315 for an annuity of $100, his investment yields him nearly 8% annually during his lifetime, and he escapes all risk of loss; but at his death the payments cease, and the capital becomes the property of the insurance company.

554. Accident Insurance is the guaranty of an indemnity in case of disability due to an accident.

555. The rates of premium for Life Insurance, as fixed by different companies, are based on the probabilities of life, determined by a table of mortality, and on the probable rates of interest which money will bear, and the probable expenses of the company.

A table of mortality shows how many persons out of a given number (as 10000), insuring at any age, may be expected to die the first, second, and third year, and so on until they all are dead.

556. The premium consists of three elements:

1. Reserve, or that portion of each premium which must be kept and improved by interest (usually *four per cent*), to pay the policy at its *certain* maturity.

2. An estimated amount for each man's share of the annual losses of the company.

3. Loading, or margin, a certain per cent of the premium to meet current expenses.

557. Most companies in this country are **Mutual**, and divide the profits among the policy holders. The profits result from the company realizing upon the reserved fund more than the assumed rate of interest, *four per cent*, from the losses by death being less than was assumed in making the premium, and from the loading or margin being more than the expenses.

Dividends are declared at the end of the first, second, third, or fourth year, and may be applied to reduce the annual premium or to increase the policy.

NOTES.—1. One half of the premium is often paid by a note, and the dividends are afterward applied toward cancelling the notes.

2. The following table rates have been selected for the different kinds of policies, for the reason that they are based on an American Table of Mortality.

3. Stock companies make no dividends to policy holders, but generally charge a rate of premium 20 to 30 per cent less than the mutual companies.

Annual Premium Rates.

For $1000 Insurance.

Age.	LIFE POLICIES. Payable at Death Only. Payments to continue for				ENDOWMENT POLICIES. Payable as Indicated, or at Death if Prior.		
	Life.	20 Years.	15 Years.	10 Years.	20 Years.	15 Years.	10 Years.
25	$20 20	$27 19	$32 15	$42 43	$47 67	$66 30	$104 35
26	20 90	27 77	32 81	43 29	47 79	66 41	104 43
27	21 50	28 36	33 51	44 19	47 92	66 51	104 52
28	22 20	28 98	34 22	45 12	48 07	66 63	104 62
29	22 80	29 63	34 97	46 08	48 22	66 76	104 72
30	23 30	30 30	35 74	47 07	48 39	66 89	104 82
31	23 90	30 99	36 53	48 10	48 57	67 02	104 94
32	24 60	31 72	37 36	49 16	48 76	67 17	105 05
33	25 30	32 47	38 22	50 26	48 98	67 33	105 18
34	26 00	33 26	39 11	51 40	49 21	67 50	105 31
35	27 10	34 08	40 03	52 58	49 47	67 69	105 44
36	27 70	34 94	40 99	53 80	49 76	67 90	105 59
37	28 60	35 84	42 00	55 06	50 08	68 13	105 74
38	29 80	36 78	43 05	56 38	50 43	68 39	105 92
39	30 60	37 78	44 14	57 74	50 83	68 68	106 12
40	31 50	38 82	45 29	59 17	51 88	69 83	107 20
41	32 60	39 93	46 50	60 66	52 38	70 21	107 47
42	33 80	41 10	47 77	62 22	52 94	70 64	107 78
43	35 00	42 34	49 12	63 85	53 58	71 13	108 14
44	36 40	43 65	50 52	65 55	54 28	71 68	108 55
45	38 00	45 03	52 00	67 32	55 06	72 29	109 01
46	39 60	46 49	53 54	69 16	55 91	72 96	109 52
47	41 20	48 02	55 15	71 07	56 84	73 69	110 08
48	43 00	49 63	56 83	73 03	57 86	74 49	110 68
49	45 00	51 33	58 58	75 07	58 97	75 37	111 35
50	47 00	53 12	60 41	77 18	60 65	76 92	112 97
51	49 00	55 43	63 06	79 99	61 99	77 97	113 76
52	51 20	57 45	65 10	82 27	63 45	79 12	114 63
53	53 80	59 59	67 23	84 64	65 04	80 38	115 58
54	56 40	61 85	69 48	87 09	66 79	81 76	116 61
55	59 40	64 26	71 84	89 64	68 69	84 25	119 15
56	62 40	67 84	75 49	94 12		85 92	120 40
57	65 60	70 61	78 17	96 94		87 76	121 78
58	69 00	73 58	81 01	99 90		89 78	123 30
59	72 70	76 75	84 03	102 99		92 00	124 98
60	76 40	80 15	87 24	106 25		94 43	126 83

[redacted]

RENEWABLE TERM POLICIES.

Age.	Annual Premium Rates for $1000 Insurance. Payable if death occurs within 10 Years	15 Years.	20 Years.
25	$12 95	$13 37	$13 82
26	13 21	13 65	14 14
27	13 49	13 94	14 49
28	13 77	14 25	14 86
29	14 08	14 58	15 27
30	14 39	14 94	15 71
31	14 73	15 32	16 20
32	15 08	15 75	16 72
33	15 44	16 21	17 29
34	15 84	16 71	17 92
35	16 27	17 26	18 60
36	16 72	17 87	19 34
37	17 26	18 54	20 15
38	17 84	19 28	21 04
39	18 49	20 09	22 01
40	19 22	20 98	23 06
41	20 03	21 97	24 22
42	20 94	23 05	25 49
43	21 94	24 24	26 88
44	23 07	25 55	28 40
45	24 29	26 98	30 04
46	25 62	28 52	31 83
47	27 06	30 19	33 75
48	28 62	32 01	35 82
49	30 30	33 96	38 05
50	32 12	36 08	40 44
51	34 14	38 38	43 01
52	36 23	40 85	45 77
53	38 53	43 53	48 73
54	41 04	46 42	51 89
55	43 76	49 54	
56	46 73	52 92	
57	49 92	56 52	
58	53 40	60 41	
59	57 18	64 60	
60	61 65		

ANNUAL POLICIES.

Age.	Rates for $100 Annuity. Payable Annually.	Annuity that $1000 will purchase. Payable Annually.
40	$1516 84	$65 93
41	1500 96	66 62
42	1484 05	67 38
43	1466 11	68 21
44	1447 16	69 10
45	1427 28	70 06
46	1406 52	71 10
47	1384 97	72 20
48	1362 62	73 39
49	1339 51	74 66
50	1315 62	76 01
51	1291 00	77 46
52	1265 66	79 01
53	1239 63	80 67
54	1212 94	82 44
55	1185 61	84 34
56	1157 67	86 38
57	1129 16	88 56
58	1100 07	90 90
59	1070 44	93 42
60	1040 32	96 12
61	1009 84	99 03
62	979 03	102 14
63	947 98	105 49
64	916 75	109 08
65	885 42	112 94
66	857 86	116 57
67	829 96	120 49
68	801 91	124 70
69	773 72	129 25
70	745 41	134 15
71	717 09	139 45
72	688 80	145 18
73	660 54	151 39
74	632 45	158 12
75	600 46	164 08
76	590 68	169 30
77	575 61	173 73
78	563 45	177 48
79	553 66	180 62
80	545 49	183 32

Examples.

558. 1. At the rate shown in the table, what sum must a man aged 33 pay annually for life for a life policy for \$7500? What sum annually for 20 years? What sum annually for an endowment policy for 15 years?

OPERATION.

$\$25.30 \times 7.5 = \$189.75$ *Ans.*

$\$32.47 \times 7.5 = \$243.53$ *Ans.*

$\$67.33 \times 7.5 = \$504.98$ *Ans.*

SOLUTION. — Consulting the table on p. 326, we find the premium on \$1000 at the age of 33 to be \$25.30 for life, \$32.47 for 20 years, and \$67.33 for a 15 years' endowment policy; and since the premiums on \$7500 will be 7.5 times those on \$1000, we multiply these sums respectively by 7.5.

2. A man insures his life for \$10000, the rate being \$21.40 per 1000. The dividend reduces the cost of the premium an average of 30%. How much is the average annual cost of his insurance?

OPERATION.

$\$21.40 \times 10 = \$214$, Premium.

30% of $\$214 = \$64.20$, Dividend.

$\$214 - \$64.20 = \$149.80$, Annual cost of Insurance *Ans.*

SOLUTION. — If \$1000 worth of insurance costs \$21.40, \$10000 worth will cost 10 times \$21.40, which is \$214, the premium. Since the dividend reduces this premium 30%, the cost of insurance will be 30% less than \$214, which is \$149.80.

NOTE. — Examples in which the premium is not stated are based on the rates given in the tables.

3. A man 30 years of age takes a life policy in a mutual company, for \$5000, the premiums continuing until death. The dividend reduces the annual premium an average of 30%. He dies after making 21 payments. How much more money will his family receive than he has paid to the company?

4. What annual premium will a man aged 36 years pay to secure an endowment policy for \$5000, payable to himself in 20 years, or to his heirs if death occurs before?

5. A young man aged 27 takes an endowment policy for \$4000, payable to himself in 20 years. If the dividend increases his policy \$2400, how much more will he receive than he has paid the company?

6. A man insured his life for \$ 25000, the rate being \$ 25 per 1000. The dividend reduced the cost of the premium an average of 10%. What is the average annual cost of his insurance?

7. If the man died 12 years after he was insured, how much would the amount received by his family exceed the total cost of insurance, no account being taken of interest?

8. If he died 4 years after he was insured, how much would the amount received by his family exceed the total cost of insurance, the interest on the money amounting to \$ 337.50?

9. A clergyman aged 45 takes an endowment policy for \$ 3000, payable to himself in 15 years, or to his family at death, and dies after making 13 payments. How much money would he have saved had he taken a policy for the same amount on the continued payment life plan?

10. A merchant aged 49 insures for \$ 8000 on the single payment life plan, and dies in the seventeenth year thereafter. How much less would his insurance have cost him had he insured on the 10 payment life plan?

11. How much must I pay at the age of 76 for an annuity of \$ 2000?

12. A has his life insured at the age of 25; B insures at the age of 35, each taking a life policy, premiums payable until death. What will be the age of each, when the amount of premium paid exceeds the face of the policy?

13. A man aged 34 insured his life for \$ 6000, payments made in 10 years. When he died there was a net gain to his family of \$ 4463.40. How many payments had he made?

14. A man aged 40 insures his life in the Conn. Mutual Life Ins. Co. for \$ 5000, premiums to continue until death. After the fourth year his premium is reduced one half by the dividend. What will be the total amount of premiums paid in thirty years?

15. A man 67 years of age purchases an annuity of \$ 602.45. How much must he pay for it?

TAXES.

559. A **Tax** is a sum of money assessed on the person or property of an individual, to meet public expenses.

560. A **Poll Tax** is a certain sum required of each male citizen liable to taxation, without regard to his property. Each person so taxed is called a *poll*.

Non-resident taxpayers are not subject to a poll tax.

561. A **Property Tax** is a sum required of each person owning property, and is always a certain *per cent* of the estimated value of his property.

Property is of two kinds, real estate and personal property.

Real Estate consists of *immovable* property, such as lands, houses, etc., and is taxed in the town or city where it is situated.

Personal Property consists of *movable* property, such as money, notes, furniture, cattle, tools, etc., and is taxed where the owner lives.

An Income Tax is a tax on annual income, salary, etc.

NOTE. — The part of a man's income or salary which is subject to taxation is usually included with his personal property.

An **Assessment Roll** is a list or schedule containing the names of all the persons liable to taxation in the place to be assessed, and the valuation of each person's taxable property.

Assessors are the officers appointed to determine the taxable value of property, to prepare the assessment rolls, and to apportion the taxes.

If the assessment includes a poll tax, then a complete list of taxable polls must also be made out.

NOTE. — In Vermont and in some other states the assessment roll is called the Grand List.

Collectors are the officers selected to receive and collect taxes.

The rate of taxation is the amount charged for each dollar of assessed valuation to raise the required amount of tax.

NOTE. — The method of raising taxes varies in different states.

562. In computations in taxes the following relations exist:

I. The *Assessed Valuation* corresponds to the *Base*.

II. The *Rate of Taxation* is the *Rate Per Cent*.

III. The *Tax* is the *Percentage*.

[redacted]

Examples.

563. To find the amount of tax.

1. In a certain town a tax of $ 4000 is to be assessed. There are 400 polls to be assessed $ .50 each, and the taxable property, as shown by the assessment roll, is valued at $ 950000. What will be the property tax on $ 1, and what will be A's tax, whose property is valued at $ 3500, and who pays for 3 polls?

OPERATION.

$$\$\ .50 \times 400 = \$\ 200, \text{ amount assessed on the polls.}$$
$$\$\,4000 - \$\,200 = \$\,3800, \text{ amt. to be assessed on property.}$$
$$\$\,3800 \div \$\,950000 = .004, \text{ rate of taxation, or } \$\,.004 = \text{the tax on } \$\,1, \textit{ Ans.}$$

$$\$\,3500 \times .004 = \$\,14, \quad \text{A's property tax;}$$
$$\$\ .50 \times 3 = \underline{\$\ 1.50}, \text{ A's poll tax;}$$
$$\$\,15.50, \text{ amount of A's tax. } \textit{Ans.}$$

RULE. — I. *Find the amount of poll tax, if any, and subtract it from the whole tax to be assessed; the remainder will be the property tax.*

II. *Divide the property tax by the whole amount of taxable property; the quotient will be the rate of taxation.*

III. *Multiply each man's taxable property by the rate of taxation, and to the product add his poll tax, if any; the result will be the whole amount of his tax.*

NOTE. — When a tax is to be apportioned among a large number of individuals, the operation is greatly facilitated by first finding the tax on $ 1, $ 2, $ 3, etc., to $ 9; then on $ 10, $ 20, $ 30, etc., to $ 90, and so on, and arranging the results as in the following table:

TABLE.

Prop.	Tax.	Prop.	Tax.	Prop.	Tax.	Prop.	Tax.
$ 1	$ .004	$ 10	$ .04	$ 100	$ .40	$ 1000	$ 4
2	.008	20	.08	200	.80	2000	8
3	.012	30	.12	300	1.20	3000	12
4	.016	40	.16	400	1.60	4000	16
5	.020	50	.20	500	2.00	5000	20
6	.024	60	.24	600	2.40	6000	24
7	.028	70	.28	700	2.80	7000	28
8	.032	80	.32	800	3.20	8000	32
9	.036	90	.36	900	3.60	9000	36

2. According to the conditions of the last example, what would be the tax of a person whose property was valued at \$ 2465, and who pays for 2 polls?

OPERATION.

From the table we find that

	The tax	on	\$ 2000		is	\$ 8.00
	"	"	" 400		"	1.60
	"	"	" 60		"	.24
	"	"	" 5		"	.02
And	"	"	" 2	polls	"	1.00
				Total tax	"	\$ 10.86 *Ans.*

3. What would A's tax be, who is assessed for \$ 8530, and who pays for 3 polls?

4. How much will C's tax be, who is assessed for \$ 987, and who pays for 1 poll?

5. The estimated expenses of a certain town for one year are \$ 6319, and the balance on hand in the public treasury is \$ 654. There are 2156 polls to be assessed at \$ .25 each, and taxable property to the amount of \$ 1864000. Besides the town tax, there is a county tax of \$ .001½ on a dollar, and a state tax of \$ .000½ on a dollar. What will be the whole amount of A's tax, whose property is valued at \$ 32560, and who pays for 3 polls?

6. What does a non-resident pay, who owns property in the same town to the amount of \$ 16840?

7. What sum must be assessed in order to raise a net amount of \$ 5561.50, and pay the commission for collecting at 2%?

NOTE.—Since the base of the collector's commission is the sum collected, the question is an example under § 524 of Percentage.

8. In a certain district a schoolhouse is to be built at an expense of \$ 9120, to be defrayed by a tax upon property valued at \$ 1536000. What must be the rate of taxation to cover both the cost of the schoolhouse and the collector's commission at 5%?

[redacted]

9. A tax of $ 13662 is to be assessed on a certain village; the property is valued at $ 1400000, and there are 2981 polls, to be taxed $ 2.00 each. What is the assessment on a dollar? What is C's tax, his property being assessed at $ 12450, and he paying for 2 polls?

10. What is the tax of a non-resident, having property in the same village valued at $ 5375?

11. A mining corporation, consisting of 30 persons, is taxed $ 4384; its property is assessed for $ 188000, and each poll is assessed $ 2.00. What per cent is their tax, and how much must he pay whose share is assessed for $ 2500, and who pays for 1 poll?

12. A tax of $ 50000 is to be assessed on a certain town; the property is valued at 3500000, and there are 5000 polls to be taxed $ 1.00 each. What will be the rate of taxation and how much will A be taxed, who owns property which is valued at $ 50000 and who pays for 3 polls?

13. The expenses of a school for one term were $ 1200 for salary of teachers, $ 57.65 for fuel, and $ 38.25 for incidentals; the money received from the school fund was $ 257.75, and the remaining part of the expense was paid by a rate bill. If the aggregate attendance was 9568 days, what was A's tax, who sent 4 pupils 46 days each?

14. The number of polls in a certain school district is 225, and the taxable property is $ 1246093.75; it is proposed to build a union schoolhouse at an expense of $ 10000. If the poll tax is $ 1.25 a poll, and the cost of collecting is $2\frac{1}{2}\%$, what will be the tax on a dollar, and what will be E's tax, who pays for 1 poll, and has property to the amount of $ 11500?

15. The expense of building a public bridge was $ 1260.52, which was defrayed by a tax upon the property of the town. The rate of taxation was $3\frac{1}{4}$ mills on one dollar, and the collector's commission was $3\frac{1}{2}\%$. What was the valuation of the property?

GENERAL AVERAGE.

564. General Average is an adjustment of losses of property at sea, when for common safety any portion is sacrificed.

565. The **Contributory Interests** are the three kinds of property which are taxed to cover the loss. These are:

1. The **Vessel**, and its value before the loss.
2. The **Freight**, less $\frac{1}{3}$ as an allowance for seamen's wages.
3. The **Cargo**, including the part sacrificed, at its market value in the port of destination.

NOTE. — In New York, Virginia, and some other states, only $\frac{1}{2}$ of the freight is made contributory to the loss.

566. The loss which is subject to general average includes:

1. **Jettison**, or property thrown overboard.
2. **Repairs** to the vessel, less $\frac{1}{3}$ on account of the superior worth of the new articles furnished.
3. **Expense of detention** to which the vessel is subject in port, including salvage, wages, and provisions of crew, adjuster's fees, etc.

567. In computations in General Average the following relations exist:

I. The *Total Contributory Interest* is the *Base*.
II. The *Loss, plus expenses*, is the *Percentage*.

Examples.

568. To find the rate of contribution and the respective contributions.

1. The ship Nelson, valued at \$ 52000, and having on board a cargo worth \$ 18000, on which the freight was \$ 3600, threw overboard goods valued at \$ 5000, to escape wreck; she then put into port, and underwent repairs amounting to \$ 1200, the expenses of detention being \$ 350. What portion of the loss will be sustained by each of the three contributory interests? What will be paid or received by the owners of the ship and freight? What by A, who owned \$ 8000 of the cargo, including \$ 3500 of the portion sacrificed, by B, who owned \$ 6000 of the cargo, including \$ 1500 of the portion sacrificed, and by C, who owned \$ 4000, or the residue of the cargo?

OPERATION.

LOSSES.		CONTRIBUTORY INTERESTS.	
Jettison	$5000	Vessel	$52000
Repairs, less ⅓	800	Freight less ⅓	2400
Cost of detention	850	Cargo	18000
Total	$6150	Total	$72400

$6150 ÷ $72400 = .0849447 +, rate per cent of loss.
$52000 × .0849447 = $4417.18, payable by vessel.
2400 × .0849447 = 203.87, " " freight.
18000 × .0849447 = 1529.00, " " cargo.
$6150.00, Total contribution.

$8000 × .0849447 = $679.56, payable by A.
6000 × .0849447 = 509.67, " " B.
4000 × .0849447 = 339.78, " " C.

$4417.18 + $203.87 = $4621.00, payable by owners of vessel and freight.
800.00 + 350.00 = 1150.00, " to " " " "
4621.00 − 1150.00 = 3471.00, balance payable by ship owners.
3500.00 − 679.56 = 2820.44, " receivable by A.
1500.00 − 509.67 = 990.33, " " " B.

RULE. — I. *Divide the sum of the losses by the sum of the contributory interests; the quotient will be the rate of contribution.*

II. *Multiply each contributory interest by the rate; the products will be the respective contributions to the loss.*

2. The ship Nevada, in distress at sea, cut away her mainmast, and cast overboard ¼ of her cargo, and then put into Havana to refit; the repairs cost $1500, and the necessary expenses of detention were $420. The ship was owned and sent to sea by George Law, and was valued at $25000; the cargo was owned by Hayden & Co., and consisted of 2800 barrels of flour, valued at $4.50 per barrel, upon which the freight was $4200. In the adjustment of the loss by general average, how much was due from Law to Hayden & Co.?

3. A coasting vessel valued at $28000, having been disabled in a storm, entered port, and was refitted at an expense of $270 for repairs, and $120 for board of seamen, pilotage and dockage. Of the cargo, valued at $5000, $2400 belonged to A, $1850 to B, and $750 to C; and the amount sacrificed for the ship's safety was $1400 of A's property, and $170 of B's; the gross charges for freight were $1500. Required the balance, payable or receivable, by each of the parties, the loss being apportioned by general average.

DUTIES OR CUSTOMS.

569. **Duties**, or **Customs**, are taxes levied on imported goods, to support the government and to protect home industry.

570. A **Tariff** is a schedule showing the rates of duties fixed by law on all kinds of imported merchandise.

571. A **Custom House** is an office established by government for the transaction of business relating to duties.

It is lawful to introduce merchandise into a country only at points where custom houses are established. A seaport town having a custom house is called a **Port of Entry**. To carry on foreign commerce secretly, without paying the duties imposed by law, is **Smuggling**.

NOTE. — Customs or duties form the principal source of revenue to the general government of the United States; by increasing the price of imported goods they operate as an indirect tax upon consumers, instead of a general direct tax.

572. Duties are of two kinds, — Ad Valorem and Specific.

Ad Valorem Duties are taxes computed on the net cost of the goods in the country from which they were imported.

Specific Duties are taxes computed on goods, without regard to their cost.

NOTE. — Most duties of the United States by the present tariff are ad valorem; a few are specific; and on some goods both specific and ad valorem duties are levied.

573. An **Invoice** is an itemized bill of goods imported, showing the quantity and cost of each kind, at the place purchased, to be submitted at the custom house as a basis for levying duties.

574. In collecting customs it is the intention of government to tax only so much of the merchandise as will be available to the importer in the market. The goods are weighed, measured, gauged, or inspected, in order to ascertain the actual quantity arrived in port; and an allowance is made in every case for waste or loss.

Tare is an allowance made for the weight of the box or the covering that contains the goods. It is ascertained, if necessary, by actually weighing one or more of the empty boxes, casks, or coverings. In common articles of importation, it is sometimes computed at a certain per cent previously ascertained by frequent tests of weighing.

Leakage is an allowance on liquors imported in casks or barrels, and is ascertained by gauging the cask or barrel in which the liquor is imported.

Breakage is an allowance on liquors imported in bottles.

Draft is an allowance for the waste of certain articles, and is made only for *statistical purposes;* it does not affect the amount of duty.

Gross Weight or **Value** is the weight or value of the goods before any allowance or discount has been made.

Net Weight or **Value** is the weight or value of the goods after all allowances have been deducted.

575. A **Bonded Warehouse** is a building in which goods on which the duties are still unpaid are stored under bond. Such goods are in the joint custody of the importer and the custom house officers.

NOTES. — 1. Imported goods left in bond, that is, in bonded warehouses, may be exported to a foreign country without payment of the duty. Goods which are left in bond become the property of the government if not removed at the end of three years.

2. The officer who superintends the collection of duties is called the *collector of the port.*

3. *Tonnage* is a tax levied on a vessel independent of its cargo, for the privilege of coming into a port of entry.

576. **Long ton** measure is employed in the custom houses of the United States in estimating goods by the ton or hundredweight.

577. In all calculations where ad valorem duties are considered: —

I. The *Net Value* of the merchandise is the worth of the net weight or quantity at the invoice price, and corresponds to the *Base.*

II. The *Duty* is computed at a certain per cent on the net value of the merchandise, and corresponds to *Percentage.*

Examples.

578. 1. What is the duty, at 40% ad valorem, on an invoice of woolen goods which cost \$ 750?

OPERATION.

$\$ 750 \times .40 = \$ 300$ *Ans.*

SOLUTION. — According to § 520, we multiply the invoice, \$ 750, which is the *base* of the duty, by the given *rate*, and obtain the duty, \$ 300.

2. What is the duty, at 24%, on 50 gross of goods, invoiced at \$1.20 per dozen, $2\frac{1}{2}\%$ being allowed for breakage?

OPERATION.

\$1.20 × 12 × 50 = \$720, Gross Value.
\$720 × .025 = \$18, Breakage.
\$720 − \$18 = \$702, Net Value.
\$702 × .24 = \$168.48, Duty *Ans.*

SOLUTION. — First we find the cost of the goods, at the invoice price, which is \$720. From this sum we deduct the allowance for breakage, \$18, and compute the duty on the remainder.

3. Having paid the duty at 8% on an invoice of goods, I find that the whole cost in store, besides freight, is \$378. What were the goods invoiced at?

OPERATION.

\$378 ÷ 1.08 = \$350 *Ans.*

SOLUTION. — According to § 523, we divide the *amount*, \$378, by 1 plus the rate, 1.08, and obtain the *base*, or invoice, \$350.

4. Find the total cost to Mr. Thompson of the following invoice of goods in United States money, and the cost per yard to him of each kind of goods.

MANCHESTER, ENG., May 1, 1895.

WILLIAM THOMPSON, NEW YORK,

Bought of WILLIAMS & SON,

20 pieces	Velveteen	No. 1,	1000 yd.	@	6*d.*
20 "	"	No. 2,	1000 "	@	7*d.*
10 "	"	No. 3,	500 "	@	12*d.*
10 "	"	No. 4,	500 "	@	15*d.*

Non-dutiable charges: commission $2\frac{1}{2}\%$, freight and transportation \$28.73. Dutiable charges: $47\frac{1}{2}\%$ ad valorem; cases and making up £4; the pound being worth \$4.90 in the market.

NOTE. — 1. It will be observed that each piece in this invoice contains 50 yd. It is customary in such invoices to state the number of pieces and the *total* number of yards instead of the number of yards in each piece. The cost of cases and making up are considered as part of the invoice, on which both commission and duty are computed.

2. The custom house valuation of the pound is always \$4.8665, and this valuation must *be taken in reckoning* the duty; but in finding the cost of the invoice the pound must be *reckoned at \$4.90,* since the importer must pay the market price for it.

OPERATION.

1000 yd. @ $6d. = 6000d.$
1000 yd. @ $7d. = 7000d.$
500 yd. @ $12d. = 6000d.$
500 yd. @ $15d. = 7500d.$
$26500d. =$

£ $110\frac{5}{12}$ + £ 4 = £ $114\frac{5}{12}$ = Invoice.

£ $114\frac{5}{12}$ @ \$ 4.90 = \$ 560.64
and £ $114\frac{5}{12}$ @ \$ 4.8665 = \$ 556.81
\$ 560.64 × .02½ = \$ 14.02, Commis.
\$ 556.81 × .47½ = \$ 264.48, Duty.

Charges:

Cost of goods,	\$ 560.64
Duty,	264.48
Commission,	14.02
Freight and transp.	28.73
	\$ 867.87, Total cost *Ans.*

\$ 867.87 ÷ 26500 = .03275, cost in U.S. money of $1d.$ worth of goods.

SOLUTION. — The goods amount to $26500d.$ or £ $110\frac{5}{12}$. Adding £ 4 for cases and making up, the invoice is £ $114\frac{5}{12}$, which at the market value of \$ 4.90 to the pound = \$ 560.64, and the commission on this at 2½ % is \$ 14.02. Since the custom house valuation of the pound is \$ 4.8665, the duty on £ $114\frac{5}{12}$ will be £ $114\frac{5}{12}$ × 4.8665 × .47½, or \$ 556.81 × .47½, which is \$ 264.48. Adding the charges, we find the total cost of the goods to be \$ 867.87.

Since $26500d.$ worth of goods cost \$ 867.87, $1d.$ worth will cost \$ 867.87 ÷ 26500 or \$ .3275; then $6d.$ worth or a yard of first kind will cost 6 times \$ .03275; a yard of the second kind, 7 times, of the third, 12 times, and of the fourth, 15 times \$ .03275.

\$ .03275 × 6 = \$ .197 a yd., cost of No. 1.
\$ .03275 × 7 = \$ .229 " " No. 2.
\$ .03275 × 12 = \$ .393 " " No. 3.
\$ .03275 × 15 = \$ .491 " " No. 4. *Ans.*

Find the duty on the following invoices of goods:

5. \$ 356 black ivory at 20% ad valorem.

6. \$ 2340 table cutlery at 35% ad valorem.

7. Twenty tons stove plates at $\frac{8}{10}$ ¢ per pound.

8. A Boston jeweler orders from Lubec a quantity of watch movements, amounting to \$ 2780. What will be the duty, at 25% ?

9. *Find the* duty at 20¢ per bu. on 2539 bu. of oil seeds.

10. The duty at 25% on an importation of white satin was $815. What was the invoice of the goods?

11. The duty on diamonds in the rough is 10%, and on cut diamonds 25%. Find the duty on an invoice of rough diamonds amounting to $1500, and polished diamonds amounting to $2758.

12. At $1\frac{1}{2}$¢ per pound, what is the duty on 600 drums of figs, each containing 14 lb., invoiced at $5\frac{1}{4}$¢ per pound?

13. The duty on an invoice of French laces at 50% was $281.25. What was the invoice of the goods?

14. An English publishing house wishing to print a book in New York, consisting of 250 pages, sends over the plates, valued at $2.50 a page. What will be the duty, electrotype plates being taxed at 25%?

15. What is the duty, at 25% ad valorem, on 100 watches invoiced at 120 francs each, the custom house valuation of the franc being $.193?

16. The duty on an invoice of $2500 worth of engravings amounted to $625. What was the rate of duty?

17. The duty on dress trimmings being 20%, find the value of an importation of such trimmings on which the duty is $512.60.

18. Find the total cost, including duty and commission, of an invoice of toys which pays $702 duty, at 30% ad valorem, the commission being $2\frac{1}{2}\%$.

19. Find the cost per yard to the American importer of 2000 yd. velveteen @ 8*d.* a yd., and 2000 yd. @ 10*d.* Non-dutiable charges: Commission, $2\frac{1}{2}\%$; freight and transportation, $38. Dutiable charges: $47\frac{1}{2}\%$ ad valorem; cases and making up, £5, the pound being worth $4.90 in the market.

20. An invoice of glass lenses amounting to $525, pays $183.75 duty. What is the rate of duty on lenses?

21. A quantity of Japan varnish, paying 25% duty, is taxed at $625. What is the amount of the invoice?

[redacted]

STOCKS AND BONDS.

579. A **Company** is an association of individuals for the prosecution of some industrial undertaking. Companies may be incorporated or unincorporated.

580. A **Corporation** is a body formed and authorized by a general law, or by a special charter, to transact business as a single individual.

581. A **Charter** is the legal act of incorporation, and defines the powers and obligations of the incorporated body.

582. A **Firm** is the name under which an unincorporated company transacts business.

NOTE. — A private banking company, or a manufacturing or commercial firm is also called a *House*.

583. The **Capital** or **Stock** is the money contributed and employed to carry on the business of an individual corporation, company, or firm; it receives different names, as Bank Stock, Railroad Stock, Government Stock, etc.

584. **Scrip** or **Certificates of Stock** are the papers or documents issued by a corporation specifying the number of shares of the joint capital which the holder owns.

585. A **Share** is one of the equal parts into which capital stock is divided. The value of a share in the original contribution of capital varies in different companies; in bank, insurance, and railroad companies of recent organization, it is usually $ 100.

586. **Stockholders** are the owners of stock, either by original title or by subsequent purchase. The stockholders constitute the company.

NOTES. — 1. The capital stock of any corporation is limited by the charter. As a general rule, only a portion is paid at the time of subscription, the residue being reserved for future outlays or disbursements.

2. When the capital stock has been all paid in, money may be raised, if necessary, by *loans*, secured by mortgage upon the property. The *bonds* (see § 587) issued for these loans entitle the holder to a fixed rate of interest.

3. Stock, as a general name, applies to the scrip and bonds of a corporation, to government bonds and public securities, and to all paper representing joint capital or claims upon corporate bodies.

4. The members of an incorporated company, under certain conditions, may be held individually liable for the debts and obligations of the company, to the amount of their interest or stock in the company, and to no greater amount. But the members of a firm or house are individually liable for all the debts and obligations of the company, without regard to the amount of their share or interest in the concern.

587. A **Bond** is a written instrument under seal, securing the payment of a sum of money at or before a specified time.

Coupon bonds have interest certificates attached to them, which are torn off as the interest becomes due. **Registered bonds** are registered in the books of the corporation issuing them, in the name of the owner.

588. **Federal** or **United States Bonds** are payable at a fixed date, and are known and quoted in commercial transactions by the rate of interest they bear.

NOTE. — Bonds may be issued by the general government or by states, cities, or towns. They are usually named from the authority that issues them, from the rate of interest they bear, and sometimes from the date on which their payment becomes due. Thus West Shore 4's are bonds issued by the West Shore Railroad Company, bearing 4% interest. U. S. 6's, 1896, are bonds issued by the United States government, bearing 6% interest, and due in 1896.

The calculations of Percentage in stocks are treated in this work under the heads of Stockjobbing, Assessments and Dividends, and Stock Investments.

STOCKJOBBING.

589. **Stockjobbing** is the buying and selling of stocks with a view to realize gain from their rise and fall in the market.

590. The **Nominal, Face,** or **Par Value** of stock is the sum for which the scrip or certificate is issued.

591. The **Market** or **Real Value** of stock is the sum for which it will sell.

592. Stock is **at par** when it sells for its nominal or par value; **above par,** at a premium or advance, when it sells for more than its nominal value; and **below par,** or at a discount, when it sells for less than its nominal value.

[redacted]

NOTE. — When the business of a company pays large profits to the stockholders, the stock will be worth more than its original cost; but when the business does not pay expenses, the value of the stock will be less than its original cost. The average market value of stock generally varies directly as the rate of profit which the business pays. Since stocks constantly fluctuate in value, the newspapers give daily quotations showing the market value of the stock for the time quoted.

593. A **Stock Broker** is a person who buys and sells stocks, as the agent of another.

594. **Brokerage** is the fee or compensation of a broker.

595. A **Stock Exchange** is an association of dealers in stocks, bonds, etc., and it is also the name of the building in which they meet to do business.

NOTES. — 1. The principal stock exchange of the United States is in New York City.

2. A man who tries to force the market price down is called a *bear*, while one who tries to force it up is called a *bull*.

596. The calculations of stockjobbing are based upon the following relations:

I. The *Par Value* of the stock is the *Base*.

II. *Premium, Discount, Brokerage,* and *Income* are each a *Percentage*, computed upon the par value.

III. The *Market Value* of stock, or the proceeds of a sale, is the *Amount* or *Difference,* according as the sum is greater or less than the par value.

IV. Quotations at a discount are 1 — the per cent, and those at a premium are 1 + the per cent.

In all examples relating to stocks $100 will be considered as a share unless otherwise specified. The market value or selling price of stocks varies from day to day, and is ascertained by consulting the **stock quotations** in the newspapers. Stocks are quoted in the papers as being at 44, 90, 105, etc.; and this means that a share, the par value of which is $100, can be bought for $44, $90, $105, etc.; or that $1 worth of the stock will cost $.44, $.90, $1.05. In the first two cases the stock is at a discount, and in the third at a premium.

NOTES. — 1. The rate of brokerage in New York City has been fixed at $\frac{1}{8}$%, but in other cities it is sometimes as high as $\frac{1}{4}$%. Since the brokerage is computed upon the par value of the stock, and has the same base as the premium or discount, it may be directly added or subtracted to the quoted rate, as the question may require.

2. When a broker *buys* stock for a customer, the brokerage must be *added* to the quoted price; and a stock quoted at 92, with brokerage $\frac{1}{8}$%, will cost the customer $92\frac{1}{8}$; *i.e.*, $\$92\frac{1}{8}$ a share, or $\$.92\frac{1}{8}$ for $1 par value. When a broker *sells* for a customer, the brokerage must be *subtracted* from the quoted price; and a stock quoted at 92 would *yield* the customer only $\$91\frac{7}{8}$ a share.

597. The following tables show the closing quotations for some stocks and bonds, each for two successive days in 1895:

STOCKS.	March 21.	March 22.	BONDS.	June 14.	June 15.
Amer. Sug. Ref. . .	97⅛	99⅛	Atchison 4's.	75⅜	75⅛
A., T., & St. F.	4¼	4⅝	B'way & 7th Av. 5's .	112	112
Balt. & Ohio	53½	53¾	E. Tenn. 5's	108	108
Cen. Pacific	17¾	17	Ill. Cen. 4's, 1953 . .	102½	102½
Chicago Gas	70¾	71¼	Kan. & Tex. 4's . . .	86	86
Chic., Mil., & St. P. .	56¾	57⅜	Mo. K. & E. 1st . . .	93	94
Del. & Hudson	128	127¾	Oregon S. L. 6's . . .	99½	99⅛
Gen. Electric	32	35	Oregon Imp. 5's . . .	47	45¼
Lake Shore	137	137½	Read. 1st pf.	32¼	32⅜
Mo. Pacific	21¼	21½	Rio G. W. 1st	77½	77¾
N. Y. Cen.	94⅞	95	So. Pac. of Ariz. 1st .	97¾	98
N. Y., Lk. E., & W. .	8⅞	9⅛	San A. & A. P. 1st 4's,	65⅛	66
No. Pacific	3⅝	3⅝	So. R'way 5's	97⅝	98½
Phil. & Read.	10⅛	9¾	Tex. Pac. 1st 5's . .	91⅞	92
Pull. Pal. Car	156	156	Un. P. D. & G. 1st . .	40½	40¼
St. Louis & So. W., pf.	11	10⅞	U. S. 4's, reg. 1907 . .	112¼	112¼
So. Pacific	17½	17⅝	U. S. 5's, coup. 1904 .	116¾	116¾
Union Pacific	8⅜	9¼	Wabash 1st	106½	106¼
U. S. Cordage, pf. . .	7¼	7⅜	West Shore 4's . . .	107	107⅛
W. U. T.	87¼	87¾	Wis. Cent. 1st	55	57

NOTE. — In the following examples where no quotations are specified, the quotations in the second column above are to be taken.

Examples.

598. **To find the value of stock at a given quotation.**

1. On March 21, 1895, a broker bought for me 32 shares of Delaware and Hudson stock, quoted at 128, charging $\frac{1}{8}\%$ brokerage. How much must I pay for it?

OPERATION.

$$\$1.28 + \$.00\tfrac{1}{8} = \$1.28\tfrac{1}{8}$$
$$3200 \times \$1.28\tfrac{1}{8} = \$4100 \textit{ Ans.}$$

SOLUTION. — \$1 worth of stock will cost \$1.28 + \$.00⅛, or \$1.28⅛. Therefore, \$3200 worth will cost 3200 × \$1.28⅛ = \$4100.

2. What shall I receive for 20 shares of Western Union Telegraph stock @ 87¾, brokerage $\frac{1}{8}\%$?

OPERATION.

$$\$.87\tfrac{3}{4} - \$.00\tfrac{1}{8} = \$.87\tfrac{5}{8}$$
$$2000 \times \$.87\tfrac{5}{8} = \$1752.50 \textit{ Ans.}$$

SOLUTION. — \$1 worth of stock will yield me \$.87¾ − \$.00⅛ = \$.87⅝. Therefore, \$2000 will yield 2000 × \$.87⅝ = \$1752.50.

RULE. — *Multiply the cost of one dollar by the number indicating the par value of the stock.*

[redacted]

Find the cost of the following stocks, using the quotations in the first column, the charge for brokerage being $\frac{1}{8}\%$.

3. 35 shares Cen. Pacific.
4. 10 shares Chicago Gas.
5. 5 shares Union Pacific.
6. 100 shares U. S. Cordage, pf.
7. 5 shares W. U. T.
8. 30 shares Mo. Pacific.
9. 50 shares N. Y. Cen.
10. 8 shares Del. & Hudson.

Using the quotations in the second column, find how much I would realize on the following stocks if I pay the broker $\frac{1}{8}\%$ for selling:

11. 10 shares Lake Shore.
12. 20 shares N. Y., Lk. E. & W.
13. 5 shares Pull. Pal. Car.
14. 25 shares S. Pacific.
15. 15 shares A., T., & S. F.
16. 10 shares Gen. Electric.
17. 100 shares No. Pacific.
18. 75 shares Phil. & Read.

19. A speculator bought on March 21, 1895, 250 shares of Amer. Sug. Ref. stock and sold the same on March 22, paying his broker $\frac{1}{8}\%$ each for buying and selling. How much did he make on the speculation?

20. Mr. Burrows bought on March 21, 1895, 50 shares General Electric stock and 100 shares Central Pacific. The next day he sold both stocks at the prices quoted. Did he gain or lose by the day's transaction, and how much?

599. To find how much stock can be purchased for a given sum.

1. I put \$62500 into the hands of a broker to be invested in Pullman Palace Car stock. How many shares shall I receive if I pay $\frac{1}{4}\%$ brokerage?

OPERATION.

$$\$1.56 + \$.00\tfrac{1}{4} = \$1.56\tfrac{1}{4}, \text{ cost of } \$1.$$
$$\$62500 \div 1.56\tfrac{1}{4} = \$40000 = 400 \text{ shares, } Ans.$$

SOLUTION. — The market value of \$1 is \$1.56; adding the brokerage, we find that every dollar's worth of stock will cost \$1.56¼. Hence for \$62500 the broker can buy \$62500 ÷ 1.56¼ = \$40000 worth, or 400 shares.

RULE. — *Divide the given sum by the cost of one dollar's worth of stock and the quotient will be the nominal amount of stock purchased.*

Brokerage being $\frac{1}{8}$%, how many shares of the following stocks could be bought on March 22, '95, for the sums stated?

2. St. Louis & So. W., pf., $ 11000.
3. Amer. Sug. Ref., $ 15880.
4. Chic., Mil., & St. Paul, $ 14375.
5. Chicago Gas, $ 57100.
6. So. Pacific, $ 92300.
7. N. Y. Central, $ 951.25.

8. My agent sells for me, at a commission of 5%, 800 bbl. flour at $ 3.50 per bbl., and invests the proceeds in railroad stock at 75$\frac{3}{4}$, brokerage $\frac{1}{4}$%. How many shares do I receive?

9. A broker exchanges $ 3600 worth of railroad bonds at 95 for 27 shares of land stock at 103, receiving the difference in cash. How much money does he receive?

10. A merchant owning 525 shares of stock worth 104 exchanges them for U. S. bonds worth 105. How much of the latter does he receive, making no deduction for brokerage?

Installments, Assessments, and Dividends.

600. An **Installment** is a portion of the capital stock required of the stockholders, as a payment on their subscription.

601. An **Assessment** is a sum required of stockholders, to meet the losses or the business expenses of the company.

NOTE. — The stock subscribed for is not always all paid at once, but *assessments* are made as the business may require. Such assessments are *installments* of the stock. Other assessments may be made to meet losses or other expenses.

602. A **Dividend** is a sum paid to the stockholders from the profits of the business.

NOTE. — When a corporation wishes to raise more money it sometimes issues additional stock and agrees to pay to the owners of such stock a specified rate of dividend before any of the profits are divided among the holders of the *common stock*. Such stock is called *preferred stock* and is usually sold at a higher price than the common stock.

603. **Gross Earnings** are all the moneys received from the regular business of the company.

604. **Net Earnings** are the moneys left after paying expenses, losses, and the interest upon the bonds, if there are any.

NOTE. — The net earnings of a corporation are usually divided among the stockholders in semiannual dividends. The income of *capital stock* is therefore fluctuating, being dependent upon the condition of business.

605. In the division of the net earnings, or the apportionments of dividends and assessments, the calculations are made by finding the rate per cent which the sum to be distributed or assessed bears to the entire capital stock. Hence,

Dividends and *Assessments* are a *Percentage* computed upon the *Par Value* of the stock as the *Base.*

Examples.

606. To find the amount of dividend.

1. A stock company declares a dividend of 6%. What does A receive, who owns 14 shares?

OPERATION.	
$ 1400 × .06 = $ 84 *Ans.*	SOLUTION. — According to § 520, we multiply the *base*, or par value, $ 1400, by the rate, .06, and obtain the dividend, $ 84.

RULE. — *Multiply the par value by the rate.*

2. A man owns 56 shares of railroad stock, and the company has declared a dividend of 8%. How much does he receive?

3. I own $ 15000 in a mutual insurance company. How many shares shall I possess after a dividend of 6% has been declared, payable in stock?

4. The net earnings of a western turnpike are $ 3616, and the amount of stock is $ 56000. If the company declares a dividend of 6%, what surplus revenue will it have?

5. The capital stock of a railroad company is $ 1830000, and its debt is $ 450000. Its gross earnings for one year were $ 407399, and its expenses $ 217621. If the company paid expenses, and interest on its debt at $5\frac{5}{8}$%, and reserved $ 78, what dividend would a stockholder receive who owned 30 shares?

6. A bank having $ 156753.19 to distribute to the stockholders, declares a dividend of $5\frac{1}{4}$%. What is the amount of its capital?

7. A ferry company, whose stock is $ 28000, pays 5% dividends semiannually. The annual expenses of the ferry are $ 2950. What are the gross earnings?

8. The passenger earnings of a western railroad in one year were \$ 574375.25, the freight and mail earnings were \$ 643672.36, the whole amount of disbursements were \$ 651113.53, and the company was able to declare a dividend of 8%. How much scrip had the company issued?

9. Having received a stock dividend of 5%, I find that I own 504 shares. How many shares had I at first?

10. I received a 6% dividend on a railroad stock, and invested the money in the same stock at 75%. My stock had then increased to \$ 16200. What was the amount of my dividend?

607. To find the rate of dividend or installment to be paid.

1. A canal company whose subscribed funds amount to \$ 84000, requires an installment of \$ 6300. What per cent must the stockholders pay?

OPERATION.

$\$ 6300 \div \$ 84000 = .07\frac{1}{2}$ *Ans.*

SOLUTION.—According to § 522, we divide the installment, \$ 6300, which is *percentage*, by the *base*, or par value \$ 84000, and obtain the rate, $.07\frac{1}{2} = 7\frac{1}{2}\%$.

RULE.—*Divide the dividend or installment by the par value of the stock.*

2. The paid-in capital of an insurance company is \$ 536000. Its receipts for one year are \$ 99280, and its losses and expenses are \$ 56400. What rate of dividend can it declare?

3. The charter of a new railroad company limits the stock to \$ 800000, of which 3 installments of 10%, 25%, and 35%, respectively, have been already paid in. The expenditures in the construction of the road have reached the sum of \$ 540000, and the estimated cost of completion is \$ 400000. If the company calls in the final installment of its stock, and assesses the stockholders for the remaining outlay, what will be the rate per cent of assessment?

4. The capital of a stock company is \$ 527000. Its losses during a certain year are \$ 7905. If the stockholders are assessed to pay this loss, what will be the rate of assessment, and how much will A pay, who owns 9 shares?

[redacted]

5. The capital of a company is $ 573000. Its receipts for one year are $ 56000 and its expenses $ 105000. If the stockholders are assessed, what will be the rate of assessment?

6. The capital stock of a certain railroad company consists of $ 200000 worth of 6% preferred stock and $ 500000 worth of common stock. If its net earnings in one year are $ 15000, what per cent of dividend can it pay on the common stock?

NOTE. — After all expenses, mortgages, etc., are paid, the specified dividend is first paid on the preferred stock, and the balance of the net earnings is paid as a dividend on the common stock.

7. The net earnings of a steamship company during a certain year were $ 36000. Its capital stock consisted of $ 400000 common stock, and $ 250000 preferred stock at 7%; besides which there was a mortgage of $ 75000 on the property of the company, bearing 5% interest. What was the rate of dividend paid on the common stock?

STOCK INVESTMENTS.

608. As we have seen the income of *capital stock* is fluctuating, but the income arising from *bonds*, whether of government or corporations, is fixed, being a certain rate per cent, semi-annually, of the par value, or face of the bonds.

Examples.

609. To find what income any investment will produce.

1. What income will be obtained by investing $ 6840 in stock bearing 6%, and purchased at 95?

OPERATION.

$$\$ 6848 \div .95 = \$ 7200, \text{ Stock purchased.}$$
$$\$ 7200 \times .06 = \$ 432, \text{ Annual Income } Ans.$$

SOLUTION. — We divide the investment, $ 6840, by the cost of $ 1, and obtain $ 7200, the stock which the investment will purchase (§ 599). And since the stock bears 6% interest, we have $ 7200 × .06 = $ 432, the annual income obtained by the investment.

RULE. — *Find how much stock the investment will purchase, and then compute the income at the given rate upon the par value.*

2. The trustees of a school invested \$50000 in U. S. 4% bonds, as a teacher's fund, purchasing the stock at $102\frac{1}{2}$. If the principal's salary is \$1000, what sum will be left to pay assistants?

3. A young man, receiving a legacy of \$48000, invested one half in 5% stock at $95\frac{1}{2}$, and the other half in 6% stock at 112, paying brokerage at $\frac{1}{4}$%. What annual income did he secure from his legacy?

4. I have \$32300 to invest, and can buy New York Central 6's at 85, or New York Central 7's at 95. How much more profitable will the latter be than the former, per year?

5. If I invest \$10504 in 5% bonds at 104, what income will my investment yield?

6. A owns a farm which rents for \$411.45 per annum. If he sells it for \$8229, and invests the proceeds in 6% bonds, at 105, paying $\frac{1}{8}$% brokerage, will his yearly income be increased or diminished, and how much?

7. A sold \$8700 of 6 per cents at 104, and invested the proceeds in 5 per cents at 94, brokerage $\frac{1}{8}$% both for selling and buying. Did he gain or lose by the exchange, and how much annually?

610. To find what sum must be invested to obtain a given income.

1. What sum must be invested in Virginia 5% bonds, purchasable at 80, to obtain an income of \$600?

OPERATION.

$$\$600 \div .05 = \$12000, \text{ stock required.}$$
$$\$12000 \times .80 = \$9600, \text{ cost or investment, } \textit{Ans.}$$

SOLUTION. — Since \$1 of the stock will obtain \$.05 income, to obtain \$600 will require $\$600 \div .05 = \$12000$. Multiplying the par value of the stock by the market price of \$1, we have $\$12000 \times .80 = \$9600$, the cost of the required stock, or the sum to be invested.

RULE. — I. *Divide the given income by the per cent which the stock pays; the quotient will be the par value of the stock required.*

II. *Multiply the par value of the stock by the market value of one dollar of the stock; the product will be the required investment.*

[redacted]

2. If Missouri State 6's are at 84, what sum must be invested in this stock to obtain an income of \$960?

3. How much must I invest in U. S. 4's, at 106, that my annual income may be \$1752?

4. If I sell \$15600 worth of railroad bonds at 97, and invest a sufficient amount of the proceeds in U. S. 4's, at 107, to yield an annual income of \$360, and buy a house with the remainder, how much will the house cost me?

5. Charles C. Thomson, through his broker, invested a certain sum of money in U. S. 4's at $107\frac{1}{4}$, and twice as much in railroad bonds (5's) at $98\frac{1}{2}$, brokerage in each case $\frac{1}{4}$%. His income from both investments was \$1644. How much did he invest in each kind of stock?

611. To find what per cent the income is of the investment, when stock is purchased at a given price.

1. What per cent of my investment shall I secure by purchasing New York 7's at 105.

OPERATION.

$.07 \div 1.05 = 6\frac{2}{3}\%$ *Ans.*

SOLUTION. — Since \$1 of the stock will *cost* \$1.05, and *pay* \$.07, the income is $\frac{7}{105} = 6\frac{2}{3}\%$ of the investment.

RULE. — *Divide the annual income by the market price of the stock; the quotient will be the rate upon the investment.*

2. What per cent of his money will a man obtain by investing in 6% stock at 108?

3. What is the rate of income upon money invested in 6% bonds, purchased at 84?

4. Which is the better investment, to buy 5's at 70, or 6's at 80?

5. Which is the more profitable, to buy 8% bonds at 120, or 5's at 75?

6. What is the rate of income upon money invested in United States 4's at 112?

7. Which is the better investment, U. S. 4's at $108\frac{1}{2}$, or railroad 5's at 98, and how much per cent per annum is it better?

8. If a man invests \$ 10000 in 5's at 98, and exchanges them at par for $7\frac{3}{10}\%$ bonds quoted at 102, what is his rate of income?

9. What per cent of his money will a man gain by investing in Pacific Railroad 6's at 105?

612. To find the price at which stock must be purchased to obtain a given rate upon the investment.

1. At what price must 6% stocks be purchased in order to obtain 8% income on the investment?

OPERATION.

$\$6 \div .08 = \$75$, *Ans.*

SOLUTION. — Since \$6, the income of \$100 of the stock, is 8% of the sum paid for it, $\$6 \div .08$, or \$75, must be the purchase price; hence the stock must be bought at 75.

RULE. — *Divide the annual income which the stock bears by the rate required on the investment; the quotient will be the price of the stock.*

2. What must I pay for 5 per cents, that my investment may yield 8%?

3. At what rate of discount must Vermont 6% bonds be purchased, that the person investing may secure $6\frac{1}{4}\%$ upon his money?

4. What rate of premium does 7% stock bear in the market when an investment pays 6%?

5. A speculator invested in a life insurance company, and received a dividend of 6%, which was $8\frac{1}{3}\%$ on his investment. At what price did he purchase?

6. What must I pay for 5 per cents, that my investment may yield 6%?

7. What rate of premium do 6 per cents bear in the market when an investment pays 5%?

8. At what rate of discount must 8 per cents be purchased, that the investment shall yield 10%?

9. What must I pay for 6 per cents, that my investment may yield 7%?

INTEREST.

SIMPLE INTEREST.

613. **Interest** is a sum paid for the use of money.

614. **Principal** is the sum for the use of which interest is paid.

615. **Rate Per Cent per annum** or **Rate of Interest** is the per cent of the principal paid for its use for one year.

Note. — The rate per cent is commonly expressed decimally as hundredths.

616. **Amount** is the sum of the principal and interest.

617. **Simple Interest** is the sum paid for the use of the principal only, during the whole time of the loan or credit.

618. **Legal Interest** is the rate per cent established by law. It varies in different states.

LEGAL INTEREST IN THE UNITED STATES IN 1898.

States and Territories.	Legal.	Special.	States and Territories.	Legal.	Special.
Alabama	8	Limit 8.	Nebraska	7	Limit 10.
Arizona	7	No limit.	Nevada	7	No limit.
Arkansas	6	Limit 10.	New Hampshire	6	Limit 6.
California	7	No limit.	New Jersey	6	Limit 6.
Colorado	8	No limit.	New Mexico	6	Limit 12.
Connecticut	6	Limit 6.	New York	6	Limit 6.
Delaware	6	Limit 6.	North Carolina	6	Limit 8.
Dist. of Columbia	6	Limit 10.	North Dakota	7	Limit 12.
Florida	8	Limit 10.	Ohio	6	Limit 8.
Georgia	7	Limit 8.	Oklahoma	7	Limit 12.
Idaho	10	Limit 18.	Oregon	8	Limit 10.
Illinois	5	Limit 7.	Pennsylvania	6	Limit 6.
Indiana	6	Limit 8.	Rhode Island	6	No limit.
Iowa	6	Limit 8.	South Carolina	7	Limit 8.
Kansas	6	Limit 10.	South Dakota	7	Limit 12.
Kentucky	6	Limit 6.	Tennessee	6	Limit 6.
Louisiana	5	Limit 8.	Texas	6	Limit 10.
Maine	6	No limit.	Utah	8	No limit.
Maryland	6	Limit 6.	Vermont	6	Limit 6.
Massachusetts	6	No limit.	Virginia	6	Limit 6.
Michigan	6	Limit 8.	Washington	8	No limit.
Minnesota	7	Limit 10.	West Virginia	6	Limit 6.
Mississippi	6	Limit 10.	Wisconsin	6	Limit 10.
Missouri	6	Limit 8.	Wyoming	12	No limit.
Montana	10	No limit.			

NOTE. — The first column represents the interest that may be legally collected if no rate is mentioned in the contract. If expressly stipulated in the contract, any interest may be charged not exceeding the limit in the second column.

619. **Usury** is a higher rate of interest than is allowed by law.

620. In the operation of interest there are five parts or elements, namely:

I. *Rate Per Cent* per annum, which is the fraction or decimal denoting how many *hundredths* of a sum of money are to be taken for a period of 1 year, and corresponds to the *Rate Per Cent* in percentage.

II. *Interest*, which is the whole sum taken for the whole period of time, whatever it may be, and corresponds to *Percentage*.

III. *Principal*, which is the *Base* or the sum on which interest is computed.

IV. *Amount*, which is the sum of principal and interest, and corresponds to *Amount* in percentage.

V. *Time*, an added element which does not appear in percentage.

These parts bear such a relation to one another, that any three of them being given (provided one of the three be the time or rate), the other two may readily be found. Hence there are many problems in Simple Interest, but the five following cases are the ones that occur most frequently in business transactions.

621. I. *Given the principal, rate, and time, to find the interest and amount.*

II. *Given the time, rate, and interest, to find the principal and amount.*

III. *Given the time, rate, and amount, to find the principal and interest.*

IV. *Given the principal, time, and interest, to find the rate.*

V. *Given the principal, rate, and interest, to find the time.*

NOTE. — It is evident that when the principal and interest are known, the *amount* may be found directly by addition; hence in problems IV and V where the principal and interest are given in the example, it is unnecessary to state the finding of the fifth element, *amount*, as part of the problem.

Examples.

622. Given the principal, rate, and time, in years and months, to find the interest and amount.

1. What are the interest and amount on $ 75.19 for 3 years 6 months, at 4% ?

OPERATION.

$$
\begin{array}{r l}
\$\,75.19 & \\
.04 & \\
\hline
\$\,3.0076 & \\
3\frac{1}{2} & \\
\hline
15038 & \\
90228 & \\
\hline
\$\,10.5266 & \text{Int. } \textit{Ans.} \\
75.19 & \\
\hline
\$\,85.7166 & \text{Amt. } \textit{Ans.}
\end{array}
$$

SOLUTION. — The interest on $ 75.19 for 1 yr., at 4%, is .04 of the principal, or $ 3.0076, and the interest for 3 yr. 6 mo. is $3\frac{1}{2}$ times the interest for 1 yr., or $ 10.5266.

The amount is equal to the sum of the principal and interest.

RULE. — I. *Multiply the principal by the rate per cent, and the product will be the interest for* 1 *year.*

II. *Multiply the product by the time in years and fractions of a year; the result will be the required interest.*

623. As the interest is the product of the principal, rate, and time, or $p \times r \times t$, we can readily express the formulas (see § 525, Percentage) for the solution of these five problems; for if $p \times r \times t = i$, if we divide both sides of the equation by $r \times t$, the result is $p = i \div (r \times t)$, etc. (Ax. 4.)

$$
\begin{aligned}
i &= p \times r \times t. \\
p &= i \div (r \times t). \\
p &= a \div (1 + r \times t). \\
a &= p \times (1 + r \times t). \\
r &= i \div (p \times t \times 1\%). \\
t &= i \div (p \times r).
\end{aligned}
$$

624. In interest, any rate per cent is confined to 1 year. Therefore, if the time is more than 1 year, the per cent will

be greater than the rate per cent per annum, and if the time is less than 1 year, the per cent will be less than the rate per cent per annum. From these facts, we deduce the following principles:

PRINCIPLES. — I. *If the rate per cent per annum is multiplied by the time, expressed in years and fractions or decimals of a year, the product will be the rate for the required time.*

II. *If the principal is multiplied by the rate for the required time, the product will be the required interest.*

III. *Interest is always the product of three factors, namely, rate per cent per annum, time, and principal.*

625. In computing interest the three factors may be taken in any order; thus, if the principal is multiplied by the rate per cent per annum, the product will be the interest for 1 year; and if the interest for 1 year is multiplied by the time expressed in years, the result will be the required interest. Hence we have the following rule:

RULE. — I. *Find the continued product of the rate per cent per annum, time, and principal, taken in such order as is most convenient; the continued product will be the required interest.*

II. *Add the principal and interest, and the result will be the amount.*

626. To find the interest on any sum, at any rate per cent for any time.

I. The interest on any sum for 1 year at 1 per cent, is .01 of that sum, and is equal to the principal with the decimal point removed two places to the left.

II. A month being $\frac{1}{12}$ of a year, $\frac{1}{12}$ of the interest on any sum for 1 year is the interest for 1 month.

III. The interest on any sum for 3 days is $\frac{3}{30} = \frac{1}{10} = .1$ of the interest for 1 month, and any number of days may readily be reduced to *tenths* of a month by dividing by 3.

IV. The interest on any sum for 1 month, multiplied by any given *time expressed* in months and tenths of a month, will produce the required interest.

[redacted]

1. What is the interest on \$ 724.68 for 2 yr. 5 mo. 19 da., at 7 % ?

OPERATION.

$$
\begin{array}{r l}
2 \text{ yr. } 5 \text{ mo. } 19 \text{ da.} = 29.6\tfrac{1}{3} \text{ mo.} & \\
12)\underline{\$\,7.2468} & \\
\$.6039 & \\
\underline{29.6\tfrac{1}{3}} & \\
2013 & \\
36234 & \\
54351 & \\
\underline{12078} & \\
\$\,17.89557 & \\
\underline{7} & \\
\$\,125.26899 & \textit{Ans.}
\end{array}
$$

SOLUTION. — We remove the decimal point in the given principal two places to the left, and have \$ 7.2468, the interest on the given sum for 1 year at 1 % (I). Dividing this by 12, we have \$ .6039, the interest for 1 month, at 1 % (II).

Multiplying this quotient by 29.6⅓, the time expressed in months and decimals of a month (III), we have \$ 17.89557, the interest on the given sum for the given time, at 1 % (IV). And multiplying this product by 7 (7 times 1 %), we have \$ 125.268, the interest on the given principal, for the given time, at the given rate per cent.

RULE. — I. To find the interest for 1 yr. at 1 %. — *Move the decimal point in the given principal two places to the left.*

II. To find the interest for 1 mo. at 1 %. — *Divide the interest for* 1 *year by* 12.

III. To find the interest for any time at 1 %. — *Multiply the interest for* 1 *month by the given time expressed in months and tenths of a month.*

IV. To find the interest at any rate per cent. — *Multiply the interest at* 1 % *for the given time by the given rate.*

CONTRACTIONS. — After removing the decimal point in the principal two places to the left, the result may be regarded either as the interest on the given principal for 12 months at 1 per cent, or for 1 month at 12 per cent. If we regard it as for 1 month at 12 per cent, and if the given rate is an aliquot part of 12 per cent, the interest on the given principal for 1 month may readily be found, by taking such an aliquot part of the interest for 1 month as the given rate is part of 12 per cent. Thus,

To find the interest for 1 month at 6 per cent, remove the decimal point two places to the left, and divide by 2.

To find it at 3 per cent, proceed as before, and divide by 4; at 4 per cent, divide by 3; at 2 per cent, divide by 6, etc.

SIX PER CENT METHOD.

627. By the table on page 353, it will be seen that the legal rate of interest in 28 states is 6 per cent. This is a sufficient reason for introducing the following brief method:

At 6% per annum the interest on $1

For 12 mo. . . . . is $.06.
" 2 mo. ($\frac{2}{12} = \frac{1}{6}$ of 12 mo.) . . " .01.
" 1 mo., or 30 da. ($\frac{1}{12}$ of 12 mo.) " $.00\frac{1}{2}$ = $.005 ($\frac{1}{12}$ of $.06).
" 6 da. ($\frac{1}{5}$ of 30 da.) . . . . . . " .001.
" 1 da. ($\frac{1}{6}$ of 6 da. = $\frac{1}{30}$ of 30 da.) is $.000\frac{1}{6}$.

PRINCIPLE. — *Hence the interest of any sum at 6% is half as many hundredths of the principal as there are months in the given time, and one sixth as many thousandths as there are days in the given time.*

628. To find the interest by the 6% method.

1. Find the interest on $500 for 2 years 4 months 7 days at 6% and at 8%.

OPERATION.

$.12	Int. of $1 for 2 yr.
.02	Int. of $1 for 4 mo.
$.001\frac{1}{6}$	Int. of $1 for 7 da.
$$.141\frac{1}{6}$	Int. of $1 for 2 yr. 4 mo. 7 da. at 6%.

Or,

2 yr. 4 mo. = 28 mo. $\frac{1}{2}$ of 28 = $.14
$\frac{1}{6}$ of 7 da. = $1\frac{1}{6}$ da. = $.001\frac{1}{6}$
$$.141\frac{1}{6}$

$$.141\frac{1}{6} \times 500 = \$70.583\frac{1}{3}$, Int. at 6% *Ans.*
8% = $\frac{8}{6}$ or $\frac{4}{3}$ of 6%.
$\frac{4}{3}$ of $$70.583\frac{1}{3} = \$94.11\frac{1}{9}$, Int. at 8% *Ans.*

SOLUTION. — Since the interest on $1 for 1 yr. is $.06, for 2 yr. it will be $.12. Since the interest for 2 mo. is $.01, for 4 mo. it will be $.02. Since the interest for 6 da. is $.001, and for 1 da. is $$.000\frac{1}{6}$, for 7 da. it is $$.001\frac{1}{6}$. Adding, we find the interest on $1 for 2 yr. 4 mo. 7 da. at 6% to be $$.141\frac{1}{6}$. Therefore, the interest on $500 is 500 times $$.141\frac{1}{6}$, which is $$70.583\frac{1}{3}$. Since 8% is $\frac{4}{3}$ of 6%, the interest at 8% is $\frac{4}{3}$ of $$70.583\frac{1}{3}$, which is $$94.11\frac{1}{9}$.

RULE. — I. *Find the interest of* \$ 1 *for the given time at* 6%, *which will be half as many hundredths of a dollar as there are months, and one sixth as many thousandths as there are days.*

II. *Multiply the principal by the interest of* \$ 1, *to obtain the interest of the given number of dollars.*

NOTES. — 1. To find the interest at any other rate per cent by this method, first find it at 6%, and then increase or diminish the result by as many times itself as the given rate is greater or less than 6%. Thus, for 7% add $\frac{1}{6}$, and for 4% subtract $\frac{1}{3}$, etc.

2. The interest of \$ 10 for 6 days, or of \$ 1 for 60 days, is \$ .01. Therefore, if the principal is less than \$ 10, and the time less than 6 days, or the principal less than \$ 1, and the time less than 60 days, the interest will be less than \$ .01, and may be disregarded.

3. Since the interest of \$ 1 for 60 days is \$ .01, the interest of \$ 1 for any number of days is as many cents as 60 is contained times in the number of days; hence if we *multiply the principal by the number of days, divide the product by 60, and point off two decimal places in the quotient, the result will be the interest in the same denomination as the principal.*

629. To find exact interest.

When the time is short, interest is usually computed for the actual number of days — 360 days being considered as a year; but when it is desirable to find the exact interest, we must consider the year as consisting of 365 days, and a leap year as of 366 days.

1. Find the *exact interest* of \$ 600 from April 6, 1893, to May 4, 1893, at 6%.

OPERATION.

28 da. = $\frac{28}{365}$ yr.

$\frac{28}{365}$ of \$ .06 = $\frac{168}{36500}$ = \$ .0046, Int. of \$ 1.

\$ .0046
600

\$ 2.76, Int. of \$ 600 *Ans.*

Or,

\$ 600 × .06 = \$ 36, Int. 1 yr. at 6%.

$\frac{28}{365}$ of \$ 36 = \$ 2.76, Int. of \$ 600 for 28 da., at 6% *Ans.*

SOLUTION. — The time is 28 days; 28 days are $\frac{28}{365}$ of a year. Since the interest on \$ 1 for 1 year is \$ .06, for $\frac{28}{365}$ of a year it is $\frac{28}{365}$ of \$ .06, which is \$ .0046. Hence the interest of \$ 600 is 600 times \$ .0046, which is \$ 2.76.

Or, since the interest of \$ 600 for *one year* at 6% is \$ 36, for $\frac{28}{365}$ of a year it is $\frac{28}{365}$ of \$ 36, which is \$ 2.76.

RULE. — I. *Express the exact number of days as the fraction of a year* (365 *days*).

II. *Find the continued product of this fraction by the rate per annum and the principal.*

Find the interest and amount on the following sums for the times given, at 6%:

2. $325 for 3 yr.
3. $1600 for 1 yr. 3 mo.
4. $36.84 for 5 mo.
5. $255 for 2 mo.
6. $35.14 for 2 yr. 9 mo. 15 da.
7. $217.15 for 3 yr. 10 mo. 1 da.
8. $721.53 for 4 yr. 1 mo. 18 da.
9. $15.125 for 15 mo. 17 da.

Find the interest and amount of the following sums for the given times at 7%:

10. $2000 for 5 yr. 6 mo.
11. $1436.59 for 2 yr. 5 mo. 18 da.
12. $520 for 5 yr. 11 mo. 29 da.
13. $100.25 for 63 da.
14. $600 for 24 da.
15. $224.14 for 8 mo. 13 da.

Find the interest and amount of the following sums at 5%:

16. $48.255 for 5 yr.
17. $750 for 1 yr. 3 mo.
18. $647.654 for 4 yr. 10 mo. 20 da.
19. $12850 for 90 da.
20. $2500 for 7 mo. 20 da.
21. $850.25 for 8 mo.
22. $48.25 for 1 yr. 2 mo. 17 da.

Find the interest and amount of the following sums at 8%:

23. $2964.12 for 11 mo.
24. $725.50 for 150 da.
25. $360 for 2 yr. 6 mo. 12 da.
26. $600 for 3 yr. 2 mo. 17 da.

Find the interest of the following sums at 10%:

27. $3045.20 for 7 mo. 15 da.
28. $1247.375 for 2 yr. 26 da.
29. $2450 for 60 da.
30. $375.875 for 3 mo. 22 da.
31. $5000 for 1 yr. 2 mo. 10 da.
32. $127.65 for 1 yr. 11 mo. 3 da.
33. What is the interest of $155.49 for 3 mo., at 6¼%?
34. What is the interest of $970.99 for 6 mo., at 5½%?
35. What is the amount of $350.50 for 2 yr. 10 mo., at 7%?
36. What is the interest of $ 95.008 for 3 mo. 24 da., at 4½%?
37. What is the amount of $145.20 for 1 yr. 9 mo. 27 da., at 12½%?
38. What is the amount of $215.34 for 4 yr. 6 mo., at 3½%?

[redacted]

39. What is the amount of $5000 for 20 da., at 7%?

40. What is the amount of $16941 for 1 yr. 7 mo., at $4\frac{3}{4}$%?

Find the exact interest of:

41. $100 for 7 yr. 7 mo., at 6%.

42. $47.50 for 4 yr. 1 mo., at 9%.

43. $2000 for 3 mo., at 7%.

44. If $1756.75 is placed at interest June 29, 1892, what amount will be due Feb. 12, 1895, at 7%?

45. If a loan of $3155.49 is made Aug. 15, 1888, at 6%, what amount will be due May 1, 1896, no interest having been paid?

46. How much is the interest on a note for $257.81, dated March 1, 1891, and payable July 16, 1893, at 7%?

47. A person borrows $3754.45, being the property of a minor who is 15 yr. 3 mo. 20 da. old. He retains it until the owner is 21 years old. How much money will then be due at 6% simple interest?

48. If a person borrows $7500 at the legal rate in Boston and lends it in Wyoming, how much will he gain in a year?

49. A man sold a piece of property for $11320; the terms were $3200 in cash on delivery, $3500 in 6 mo., $2500 in 10 mo., and the remainder in 1 yr. 3 mo., with 7% interest. What was the whole amount to be paid?

50. If a man borrows $15000 in Idaho, and lends it in New York, how much will he lose in 146 days?

51. Hubbard & Northrop bought bills of dry goods of Bowen, McNamee & Co., New York, as follows: July 15, 1895, $1250; Oct. 4, 1895, $3540.84; Dec. 1, 1895, $575; and Jan. 24, 1896, $816.90. They bought on time, paying legal interest. How much was the whole amount of their indebtedness, March 1, 1896?

52. Mr. Gordon bought a piece of property of Mr. Hilton on Jan. 1, 1896 for $2870, and agreed to pay for it in 1 yr. 6 mo., with interest at $6\frac{1}{2}$%. How much was due Mr. Hilton *on July 1, 1897?*

53. A broker allows 6% per annum on all moneys deposited with him. If on an average he lends out every $ 100 received on deposit 11 times during the year, for 33 days each time at 2% a month, how much does he gain by interest on $ 1000?

54. A man engaged in business with a capital of $ 21840, is making $12\frac{1}{2}$% per annum on his capital; but on account of ill health he quits his business, and loans his money at $7\frac{3}{4}$%. How much does he lose in 2 yr. 5 mo. 10 da. by the change?

55. A speculator wishing to purchase a tract of land containing 450 acres at $ 27.50 an acre, borrows the money at $5\frac{1}{2}$%. At the end of 4 yr. 11 mo. 20 da. he sells $\frac{2}{3}$ of the land at $ 34 an acre, and the remainder at $ 32.55 an acre. How much does he lose by the transaction?

630. Given, the time, rate per cent, and interest, to find the principal and amount.

1. What sum of money will produce $ 87.42 interest in 4 years, at 6%, and what will be the amount?

OPERATION.

$ .24, Int. of $ 1 for 4 yr. at 6 %.

$$\left.\begin{array}{l}\$\,87.42 \div .24 = \$\,364.25,\ \text{Prin.} \\ \$\,364.25 + \$\,87.42 = \$\,451.67,\ \text{Amt.}\end{array}\right\}\ \textit{Ans.}$$

SOLUTION. — Since $ .24 is the interest of $ 1 for 4 years at 6%, $ 87.42 must be the interest of as many dollars, for the same time and at the same rate, as $ .24 is contained times in $ 87.42. Dividing, we obtain $ 364.25, the required principal, and adding the principal and interest we obtain the amount, $ 451.67.

RULE. — I. *Divide the given interest by the interest of $ 1 for the given time at the given rate. The result will be the principal.*

II. *Add the principal and interest to find the amount.*

2. What sum of money, invested at $6\frac{1}{2}$%, will produce $ 279.825 in 1 yr. 6 mo.?

3. What sum will produce $ 63.75 interest in 6 mo. 24 da. at $7\frac{1}{2}$%?

4. What sum of money will produce $ $12\frac{1}{2}$ interest in 10 days at 10%?

5. What sum must be invested in real estate paying $12\frac{1}{2}\%$ profit in rents, to give an income of \$ 3125?

6. What is the value of a house and lot that pays a profit of $9\frac{1}{2}\%$ by renting it at \$ 30 per month?

7. What sum must be invested in real estate yielding 8% profit to produce an income of \$ 5615?

8. What sum of money, put at interest for 6 yr. 5 mo. 11 da., at 7%, will gain \$ 3159.14?

631. Given, the time, rate per cent, and amount, to find the principal and interest.

1. What sum of money in 2 years 6 months, at 7 %, will amount to \$ 136.535, and what will be the interest?

OPERATION.

\$ 1.175, Amt. of \$ 1 for 2 yr. 6 mo. at 7 %.
\$ 136.535 ÷ 1.175 = \$ 116.20, Prin. } *Ans.*
\$ 136.535 − \$ 116.20 = \$ 20.335, Int. }

SOLUTION.—Since \$ 1.175 is the amount of \$ 1 for 2 years 6 months, at 7%, \$ 136.535 must be the amount of as many dollars, for the same time and at the same rate, as 1.175 is contained times in \$ 136.535. Dividing, we obtain \$ 116.20, the required principal, and subtracting this principal from the amount we obtain \$ 20.335, the interest.

RULE.—I. *Divide the given amount by the amount of* \$ 1 *for the given time at the given rate to find the principal.*

II. *Subtract the principal from the amount to find the interest.*

NOTE.—As the amount of \$ 1 is always expressed by the same number as 1 + the rate for the given time, we may divide the amount by 1 + the rate by the time, *i.e.*, $p = a \div (1 + r \times t)$ (§ 628).

2. What principal in 2 yr. 3 mo. 10 da., at 5 per cent, will amount to \$ $1893.61\frac{1}{8}$?

3. What sum put at interest at $3\frac{1}{2}\%$ for 10 yr. 2 mo., will amount to \$ 15660?

4. What is the interest of that sum for 2 yr. 8 mo. 29 da., at 7 %, which at the same time and rate, will amount to \$ 1568.97?

5. If the time is 3 yr., the rate 4%, and the amount *\$ 31826.40,* find the interest.

6. What is the interest of that sum for 243 days at 8%, which at the same time and rate, will amount to \$11119.70?

7. What sum put at interest at $5\frac{1}{4}$% for 8 yr. 5 mo., will amount to \$1897.545?

8. What sum of money put at interest at 4% for 3 yr. 6 mo. 18 da., will amount to \$31119.50?

632. Given, the principal, time, and interest, to find the rate per cent.

1. I received \$315 for 3 years' interest on a mortgage of \$1500. What was the rate per cent?

OPERATION.

$$\begin{array}{r} \$\,15.00 \\ 3 \\ \hline \$\,45.00 \end{array}\text{, Int. for 3 yr. at 1\%.}$$

\$315 ÷ \$45 = 7; 7% *Ans.*

SOLUTION.—Since \$45 is the interest on the mortgage for 3 years at 1 per cent, \$315 must be the interest on the mortgage for the same time, at as many times 1 per cent as \$45 is contained times in \$315. Dividing, we obtain 7, the required rate per cent.

RULE.—*Divide the given interest by the interest on the principal for the given time at* 1 *per cent.*

2. If I loan \$750 at simple interest, and at the end of 1 yr. 3 mo. receive \$796.87$\frac{1}{2}$, what is the rate per cent?

3. If I pay \$10.58 for the use of \$1700 for 28 days, what is the rate of interest?

4. I borrowed \$600, and at the end of 9 yr. 6 mo. returned \$856.50. What was the rate per cent?

5. A man invests \$7266.28, which gives him an annual income of \$744.7937. What rate of interest does he receive?

6. If C buys stock at 70, and every 6 months receives a dividend of 4%, what annual rate of interest does he receive?

7. At what rate per annum of simple interest will any sum of money double itself in 4, 6, 8, and 10 years, respectively?

SUGGESTION.—To double itself a sum must gain 100%. At 1% interest it would take 100 years to gain 100%, at 4% it would take 100 ÷ 4 = 25 years, etc.

8. At what rate per annum of simple interest will any sum treble itself in 2, 5, 7, 12, and 20 years respectively?

SUGGESTION.—To treble itself a sum must gain 200%.

9. A house that rents for \$760.50 per annum, cost \$7800. What per cent does it pay on the investment?

633. Given the principal, rate per cent, and interest, to find the time.

1. In what time will \$924 gain \$151.536, at 6%?

OPERATION.

$$
\begin{array}{l}
\$924 \\
\underline{\quad .06} \\
\$55.44, \text{Int. of } \$924 \text{ for 1 yr. at } 6\%. \\
\$151.536 \div \$55.44 = 2.73\tfrac{1}{3}. \\
2.73\tfrac{1}{3} \text{ yr.} = 2 \text{ yr. } 8 \text{ mo. } 24 \text{ da. } \textit{Ans.}
\end{array}
$$

SOLUTION.—Since \$55.44 is the interest of \$924 for 1 year at 6%, \$151.536 must be the interest of the same sum, at the same rate, for as many years as \$55.44 is contained times in \$151.536, which is 2.73⅓ times. Reducing the mixed decimal to its equivalent compound number we have 2 yr. 8 mo. 24 da., the required time.

RULE.—*Divide the given interest by the interest on the principal for* 1 *year; the quotient will be the required time in years and decimals.*

2. In what time will \$273.51 amount to \$312.864, at 7%?

3. How long must \$650.82 be on interest to amount to \$761.44, at 5%?

4. How long must \$204 be on interest to amount to \$217.09, at 7%?

5. How long may I borrow \$750, with interest at 6%, so that it may amount to \$942?

6. How long will it take any sum of money to double itself by simple interest at 3, 4½, 6, 7, and 10%? How long to quadruple itself?

SUGGESTION.—At 3% a sum will gain 100% in 100 ÷ 3 = 33⅓ years.

7. In what time will \$9750 produce \$780 interest at 12%?

8. Mr. Cook loaned \$1600 at 6% until it amounted to \$2000. What was the time?

9. I borrowed \$2250 of a friend at 5%, and kept it until it amounted to \$2643.75. When did I repay the debt?

ANNUAL INTEREST.

634. When money is loaned for more than one year, the interest is usually made payable annually or semiannually by the terms of the contract. If a contract reads "*with interest payable annually*" and the interest is not paid when due, some states allow the creditors to collect simple interest on each year's interest from the time it is due to the date of settlement.

635. Annual Interest is the process of computing simple interest on the principal and on each year's interest not paid when due.

NOTE. — 1. The term *annual interest* is used for *total interest* including simple interest and interest on unpaid interest, and also in another sense to indicate *yearly* interest, *i.e.* the interest due at the end of each year. (See p. 379, line 12.)

2. The int. on the unpaid int. must be so computed as not to increase the original principal. This is the difference between total annual int. and compound int. (§ 637).

Examples.

636. To find interest, payable annually.

1. Find the interest and amount of $ 8000 for 5 yr., at 6 %, interest payable annually.

OPERATION.

Int. of $ 8000 for 1 yr. at 6 % = $ 480.
" " $ 8000 for 5 yr. at 6 % = $ 2400.
" " $ 480 for 10 yr. at 6 % = $ 288.

$$\left.\begin{array}{l}\$\,2400 + \$\,288 = \$\,2688, \text{ Total Ann. Int.} \\ \$\,8000 + \$\,2688 = \$\,10688, \text{ Amount}\end{array}\right\} \textit{Ans.}$$

SOLUTION. — The interest on $ 8000 for 1 yr. at 6 % is $ 480, and for 5 yr. it is $ 2400.

The interest for the first year, remaining unpaid, draws interest for 4 yr.; that for the second year, for 3 yr.; that for the third year, for 2 yr.; and that for the fourth year, for 1 yr., the sum of which is equal to the interest of $ 480 for 4 yr. + 3 yr. + 2 yr. + 1 yr. = 10 yr.; and the interest of $ 480 at 6 % for 10 yr. is $ 288. Hence the total interest is $ 2400 + $ 288, or $ 2688, and the amount is $ 10688.

RULE. — I. *Compute the interest on the principal for the given time and rate, to which add the interest on each year's interest for the time it has remained unpaid.*

II. *To obtain the latter, when the interest has remained unpaid for a number of years, multiply the interest for one year by the product of the number of years and half that number diminished by one.*

Thus, if the time is 9 yr., the interest for 1 yr. should be multiplied by $9 \times (9 - 1) \div 2$, or $9 \times 4 = 36$. Since the interest for the first year draws 8 years' interest, that for the second 7 years' interest, etc., the sum of the series $8 + 7 + 6 + 5 + 4 + 3 + 2 + 1$ is 36.

2. What is the total interest of $1500 for 4 yr. at 7%, payable annually?

3. What will $3500 amount to in 10 yr., with interest annually, at 8%?

4. What is the difference between the simple interest of $2500 for 6 yr. at 6%, and the interest if payable annually?

5. Find the amount of $575, at 8% annual int., for 9 yr.

6. Find the amount of $2300 for 3 yr. 5 mo. 18 da., at 8%, with interest payable annually.

7. Find the amount of $1800 for 6 yr., at 6%, with interest payable annually.

8. What is the difference between the simple interest of $1800 for 5 yr. 8 mo., at 5%, and the interest if payable annually?

9. Find the total interest and the amount of $10000 for 3 yr. 6 mo., at 8%, payable annually.

10. Find the amount of $3420 for 6 yr. 6 mo., at 6%, with interest annually.

11. Find the total interest and the amount of $850 for 4 yr. 2 mo., at 7%, with interest annually.

12. What is the difference between the total interest payable annually and the simple interest of $5600 for 4 yr. 3 mo. 15 da., at 6%?

13. What is the interest of $10000 for 3 yr. 5 mo. 10 da., at 6%, payable annually? What is the simple interest? What is the difference between the former and the latter?

COMPOUND INTEREST.

637. Compound Interest is interest on both principal and interest, when the interest is not paid when due.

NOTES.—1. The simple interest may be added to the principal annually, semiannually, or quarterly, as the parties may agree; but the taking of compound interest is not legal.

2. In *annual interest* the additional int. is computed only on the unpaid simple int.; but in *compound* interest the principal is increased at the end of each interval by all the interest accrued during that interval. Hence compound int. is more than annual int.

638. The problems in compound interest are the same as those in simple interest (§ 621).

I. Given the principal, rate, and time, to find the compound interest and amount (§ 622).

II. Given the time, rate, and compound interest, to find the principal and amount (§ 630).

III. Given the time, rate, and amount, to find the principal and compound interest (§ 631).

IV. Given the principal, time, and compound interest or amount, to find the rate (§ 632).

V. Given the principal and rate, and compound interest or amount, to find the time (§ 633).

Examples.

639. Given the principal, rate, and time, to find the compound interest and amount.

1. What is the compound interest of $640 for 4 years at 5%?

FIRST OPERATION.

	$640	Principal for 1st year.
$640 × 1.05 =	$672	" " 2d "
$672 × 1.05 =	$705.60	" " 3d "
$705.60 × 1.05 =	$740.88	" " 4th "
$740.88 × 1.05 =	$777.92	Amount " 4 years.
	640.	Given Principal.
	$137.92	Compound Interest *Ans.*

SECOND OPERATION.

$1.05 \times 1.05 \times 1.05 \times 1.05 = 1.2155.$

$640 × 1.2155 =	$777.92	Amount for 4 years.
	640.00	Principal.
	$137.92	Compound Interest *Ans.*

SOLUTION. — The amount for the first year will be 105% of the principal; for the second year, 105% of this result; for the third year, 105% of the amount for the second year; and for the fourth year, 105% of the amount for the third year. The amount for the fourth year will therefore be $1.05 \times 1.05 \times 1.05 \times 1.05 = 1.2155$ of the principal; $\$640 \times 1.2155 = \$777.92$. Subtracting the principal from this amount, we find the compound interest to be \$137.92.

RULE. — I. *Find the amount of the given principal at the given rate for one interval, and make it the principal for the second interval.*

II. *Find the amount of this new principal, and make it the principal for the third interval, and so continue for the given number of intervals.*

III. *Subtract the given principal from the last amount; the remainder will be the compound interest.* Or,

I. *Find the amount for the given time expressed in per cent, and take this per cent of the principal. The result will be the amount.*

II. *Subtract the principal from this amount to find the compound interest.*

NOTE.—By the use of the compound interest table on pp. 370, 371, the second method may be greatly abbreviated. When the interest is payable semiannually or quarterly, find the amount of the given principal for the first interval, and make it the principal for the second interval, proceeding in all respects as when the interest is payable yearly. When the time contains years, months, and days, find the amount for the years, upon which compute the interest for the months and days, and add it to the last amount, before subtracting.

2. What is the amount of \$300 for 4 years at 6% compound interest, payable semiannually?

OPERATION.

$$\begin{array}{r} \$1.26677 \\ 300 \\ \hline \$380.031 \end{array}$$ *Ans.*

SOLUTION. — The amount of \$1 at 6%, comp. int. payable semiannually, is the same as the amount of \$1 at 3%, payable *annually* for twice as many years. We therefore take, from the table, the amount of \$1 for 8 yr. at 3%, and multiply this amt. by the given principal.

Find the compound interest and amount of:

3. \$750, 4 yr., 6%. 4. \$250, 3 yr., 7%.

5. \$376, 3 yr. 8 mo. 15 da., 6%.

6. \$1475.50, $2\frac{1}{2}$ yr., 7%, compounded semiannually.

7. \$1840, 1 yr. 10 mo. 20 da., 8%, compounded quarterly.

8. What is the amount of \$536.75 for 12 yr., at 8%, compound interest?

AMOUNT OF $1 AT COMPOUND INTEREST IN ANY NUMBER OF YEARS, NOT EXCEEDING FIFTY-FIVE.

Yr.	2 per cent.	2½ per cent.	3 per cent.	3½ per cent.	4 per cent.	4½ per cent.
1	1.0200 0000	1.0250 0000	1.0300 0000	1.0350 0000	1.0400 0000	1.0450 0000
2	1.0404 0000	1.0506 2500	1.0609 0000	1.0712 2500	1.0816 0000	1.0920 2500
3	1.0612 0800	1.0768 9062	1.0927 2700	1.1087 1787	1.1248 6400	1.1411 6612
4	1.0824 3216	1.1038 1289	1.1255 0881	1.1475 2300	1.1698 5856	1.1925 1860
5	1.1040 8080	1.1314 0821	1.1592 7407	1.1876 8631	1.2166 5290	1.2461 8194
6	1.1261 6242	1.1596 9342	1.1940 5230	1.2292 5533	1.2653 1902	1.3022 6012
7	1.1486 8567	1.1886 8575	1.2298 7387	1.2722 7926	1.3159 3178	1.3608 6183
8	1.1716 5938	1.2184 0290	1.2667 7008	1.3168 0904	1.3685 6905	1.4221 0061
9	1.1950 9257	1.2488 6297	1.3047 7318	1.3628 9735	1.4233 1181	1.4860 9514
10	1.2189 9442	1.2800 8454	1.3439 1638	1.4105 9876	1.4802 4428	1.5529 6942
11	1.2433 7431	1.3120 8666	1.3842 3387	1.4599 6972	1.5394 5406	1.6228 5305
12	1.2682 4179	1.3448 8882	1.4257 6089	1.5110 6866	1.6010 3222	1.6958 8143
13	1.2936 0663	1.3785 1104	1.4685 3371	1.5639 5606	1.6650 7351	1.7721 9610
14	1.3194 7876	1.4129 7382	1.5125 8972	1.6186 9452	1.7316 7645	1.8519 4492
15	1.3458 6834	1.4482 9817	1.5579 6742	1.6753 4883	1.8009 4351	1.9352 8244
16	1.3727 8570	1.4845 0562	1.6047 0644	1.7339 8604	1.8729 8125	2.0223 7015
17	1.4002 4142	1.5216 1826	1.6528 4763	1.7946 7555	1.9479 0050	2.1133 7681
18	1.4282 4625	1.5596 5872	1.7024 3306	1.8574 8920	2.0258 1652	2.2084 7877
19	1.4568 1117	1.5986 5019	1.7535 0605	1.9225 0132	2.1068 4918	2.3078 6031
20	1.4859 4740	1.6386 1644	1.8061 1123	1.9897 8886	2.1911 2314	2.4117 1402
21	1.5156 6634	1.6795 8185	1.8602 9457	2.0594 3147	2.2787 6807	2.5202 4116
22	1.5459 7967	1.7215 7140	1.9161 0341	2.1315 1158	2.3699 1879	2.6336 5201
23	1.5768 9926	1.7646 1068	1.9735 8651	2.2061 1448	2.4647 1555	2.7521 6635
24	1.6084 3725	1.8087 2595	2.0327 9411	2.2833 2849	2.5633 0417	2.8760 1383
25	1.6406 0599	1.8539 4410	2.0937 7793	2.3632 4498	2.6658 3633	3.0054 3446
26	1.6734 1811	1.9002 9270	2.1565 9127	2.4459 5856	2.7724 6979	3.1406 7901
27	1.7068 8648	1.9478 0002	2.2212 8901	2.5315 6711	2.8833 6858	3.2820 0956
28	1.7410 2421	1.9964 9502	2.2879 2678	2.6201 7196	2.9987 0332	3.4296 9999
29	1.7758 4469	2.0464 0739	2.3565 6551	2.7118 7798	3.1186 5145	3.5840 3649
30	1.8113 6158	2.0975 6758	2.4272 6247	2.8067 9370	3.2433 9751	3.7453 1813
31	1.8475 8882	2.1500 0677	2.5000 8035	2.9050 3148	3.3731 3341	3.9138 5745
32	1.8845 4059	2.2037 5694	2.5750 8276	3.0067 0759	3.5080 5875	4.0899 8104
33	1.9222 3140	2.2588 5086	2.6523 3524	3.1119 4235	3.6483 8110	4.2740 3018
34	1.9606 7603	2.3153 2213	2.7319 0530	3.2208 6033	3.7943 1634	4.4663 6154
35	1.9998 8955	2.3732 0519	2.8138 6245	3.3335 9045	3.9460 8899	4.6673 4781
36	2.0398 8734	2.4325 3532	2.8982 7833	3.4502 6611	4.1039 3255	4.8773 7846
37	2.0806 8509	2.4933 4870	2.9852 2668	3.5710 2543	4.2680 8986	5.0968 6049
38	2.1222 9879	2.5556 8242	3.0747 8348	3.6960 1132	4.4388 1345	5.3262 1921
39	2.1647 4477	2.6195 7448	3.1670 2698	3.8253 7171	4.6163 6599	5.5658 9908
40	2.2080 3966	2.6850 6384	3.2620 3779	3.9592 5972	4.8010 2063	5.8163 6454
41	2.2522 0046	2.7521 9043	3.3598 9893	4.0978 3381	4.9930 6145	6.0781 0094
42	2.2972 4447	2.8209 9520	3.4606 9589	4.2412 5799	5.1927 8391	6.3516 1548
43	2.3431 8936	2.8915 2008	3.5645 1677	4.3897 0202	5.4004 9527	6.6374 3818
44	2.3900 5314	2.9638 0808	3.6714 5227	4.5433 4160	5.6165 1508	6.9361 2290
45	2.4378 5421	3.0379 0328	3.7815 9584	4.7023 5855	5.8411 7568	7.2482 4843
46	2.4866 1129	3.1138 5086	3.8950 4372	4.8669 4110	6.0748 2271	7.5744 1961
47	2.5363 4351	3.1916 9713	4.0118 9503	5.0372 8404	6.3178 1562	7.9152 6849
48	2.5870 7039	3.2714 8956	4.1322 5188	5.2135 8898	6.5705 2824	8.2714 5557
49	2.6388 1179	3.3532 7680	4.2562 1944	5.3960 6459	6.8333 4937	8.6436 7107
50	2.6915 8803	3.4371 0872	4.3839 0602	5.5849 2686	7.1066 8335	9.0326 3627
51	2.7454 1979	3.5230 3644	4.5154 2320	5.7803 9930	7.3909 5068	9.4391 0490
52	2.8003 2819	3.6111 1235	4.6508 8590	5.9827 1327	7.6865 8871	9.8638 6463
53	2.8563 3475	3.7013 9016	4.7904 1247	6.1921 0824	7.9940 5226	10.3077 3853
54	2.9134 6144	3.7939 2491	4.9341 2485	6.4088 3202	8.3138 1435	10.7715 8677
55	2.9717 3067	3.8887 7303	5.0821 4859	6.6331 4114	8.6463 6692	11.2563 0817

[redacted]

Amount of $1 at Compound Interest in any number of years, not exceeding fifty-five.

Yr.	5 per cent.	6 per cent.	7 per cent.	8 per cent.	9 per cent.	10 per cent.
1	1.0500 000	1.0600 000	1.0700 000	1.0800 000	1.0900 000	1.1000 000
2	1.1025 000	1.1236 000	1.1449 000	1.1664 000	1.1881 000	1.2100 000
3	1.1576 250	1.1910 160	1.2250 430	1.2597 120	1.2950 290	1.3310 000
4	1.2155 063	1.2624 770	1.3107 960	1.3604 890	1.4115 816	1.4641 000
5	1.2762 816	1.3382 256	1.4025 517	1.4693 281	1.5386 240	1.6105 100
6	1.3400 956	1.4185 191	1.5007 304	1.5868 743	1.6771 001	1.7715 610
7	1.4071 004	1.5036 303	1.6057 815	1.7138 243	1.8280 391	1.9487 171
8	1.4774 554	1.5938 481	1.7181 862	1.8509 302	1.9925 626	2.1435 888
9	1.5513 282	1.6894 790	1.8384 592	1.9990 046	2.1718 933	2.3579 477
10	1.6288 946	1.7908 477	1.9671 514	2.1589 250	2.3673 637	2.5937 425
11	1.7103 394	1.8982 986	2.1048 520	2.3316 390	2.5804 264	2.8531 167
12	1.7958 563	2.0121 965	2.2521 916	2.5181 701	2.8126 648	3.1384 284
13	1.8856 491	2.1329 283	2.4098 450	2.7196 237	3.0658 046	3.4522 712
14	1.9799 316	2.2609 040	2.5785 342	2.9371 936	3.3417 270	3.7974 983
15	2.0789 282	2.3965 582	2.7590 315	3.1721 691	3.6424 825	4.1772 482
16	2.1828 746	2.5403 517	2.9521 638	3.4259 426	3.9703 059	4.5949 730
17	2.2920 183	2.6927 728	3.1588 152	3.7000 181	4.3276 334	5.0544 703
18	2.4066 192	2.8543 392	3.3799 323	3.9960 195	4.7171 204	5.5599 173
19	2.5269 502	3.0255 995	3.6165 275	4.3157 011	5.1416 613	6.1159 090
20	2.6532 977	3.2071 355	3.8696 845	4.6609 571	5.6044 108	6.7275 000
21	2.7859 626	3.3995 636	4.1405 624	5.0338 337	6.1088 077	7.4002 499
22	2.9252 607	3.6035 374	4.4304 017	5.4365 404	6.6586 004	8.1402 749
23	3.0715 238	3.8197 497	4.7405 299	5.8714 637	7.2578 745	8.9543 024
24	3.2250 999	4.0489 346	5.0723 670	6.3411 807	7.9110 832	9.8497 327
25	3.3863 549	4.2918 707	5.4274 326	6.8484 752	8.6230 807	10.8347 059
26	3.5556 727	4.5493 830	5.8073 529	7.3963 532	9.3991 579	11.9181 765
27	3.7334 563	4.8223 459	6.2138 676	7.9880 615	10.2450 821	13.1099 942
28	3.9201 291	5.1116 867	6.6488 384	8.6271 064	11.1671 395	14.4209 936
29	4.1161 356	5.4183 879	7.1142 571	9.3172 749	12.1721 821	15.8630 930
30	4.3219 424	5.7434 912	7.6122 550	10.0626 569	13.2676 785	17.4494 023
31	4.5380 395	6.0881 006	8.1451 129	10.8676 694	14.4617 695	19.1943 425
32	4.7649 415	6.4533 867	8.7152 708	11.7370 830	15.7633 288	21.1137 768
33	5.0031 885	6.8405 899	9.3253 398	12.6760 496	17.1820 284	23.2251 544
34	5.2533 480	7.2510 253	9.9781 135	13.6901 336	18.7284 109	25.5476 699
35	5.5160 154	7.6860 868	10.6765 815	14.7853 443	20.4139 679	28.1024 369
36	5.7918 161	8.1472 520	11.4239 422	15.9681 718	22.2512 250	30.9126 805
37	6.0814 069	8.6360 871	12.2236 181	17.2456 256	24.2538 353	34.0039 486
38	6.3854 773	9.1542 524	13.0792 714	18.6252 756	26.4366 805	37.4043 434
39	6.7047 512	9.7035 075	13.9948 204	20.1152 977	28.8159 817	41.1447 778
40	7.0399 887	10.2857 179	14.9744 578	21.7245 215	31.4094 200	45.2592 556
41	7.3919 882	10.9028 610	16.0226 699	23.4624 832	34.2362 679	49.7851 811
42	7.7615 876	11.5570 327	17.1442 568	25.3394 819	37.3175 320	54.7636 992
43	8.1496 669	12.2504 546	18.3443 548	27.3666 404	40.6761 098	60.2400 692
44	8.5571 503	12.9854 819	19.6284 596	29.5559 717	44.3369 597	66.2640 761
45	8.9850 078	13.7646 108	21.0024 518	31.9204 494	48.3272 861	72.8904 837
46	9.4342 582	14.5904 875	22.4726 234	34.4740 853	52.6767 419	80.1795 321
47	9.9059 711	15.4659 167	24.0457 070	37.2320 122	57.4176 486	88.1974 853
48	10.4012 697	16.3938 717	25.7289 065	40.2105 731	62.5852 370	97.0172 338
49	10.9213 331	17.3775 040	27.5299 300	43.4274 190	68.2179 083	106.7189 572
50	11.4673 998	18.4201 543	29.4570 251	46.9016 125	74.3575 201	117.3908 529
51	12.0407 698	19.5253 635	31.5190 168	50.6537 415	81.0496 969	129.1299 382
52	12.6428 083	20.6968 853	33.7253 480	54.7060 408	88.3441 696	142.0429 320
53	13.2749 487	21.9386 985	36.0861 224	59.0825 241	96.2951 449	156.2472 252
54	13.9386 961	23.2550 204	38.6121 509	63.8091 260	104.9617 079	171.8719 477
55	14.6356 309	24.6503 216	41.3150 015	68.9138 561	114.4082 616	189.0591 425

640. Given the time, rate, and compound interest, to find the principal and amount.

1. What principal at 2% compound interest will gain \$1090.80 in 2 years, and what will be the amount?

OPERATION.

$$\begin{array}{lll} \$1.0404 - \$1 & = \$.0404, & \text{Comp. Int. } \$1 \text{ for 2 yr. at } 2\%. \\ \$1090.80 \div .0404 & = \$27000, & \text{Prin.} \\ \$27000 + \$1090.80 & = \$28090.80, & \text{Amt.} \end{array} \Bigg\} \textit{Ans.}$$

SOLUTION. — By the table we find the amount of \$1 for 2 yr. at 2% to be \$1.0404, hence the compound interest for \$1 is \$.0404, and \$1090.80 is the compound interest of as many dollars as .0404 is contained times in \$1090.80. The amount is the principal plus the interest.

RULE. — I. *Divide the given compound interest by the compound interest of \$1 for the given rate; the result will be the principal.*

II. *Add the principal and compound interest to find the amount.*

What principal at compound interest will gain:

2. \$5314.80 in 2 yr. at 6%? 3. \$775.215 in 2 yr. at 7%?

4. If \$8851.38 comp. int. is paid for a sum for 4 yr. at $3\frac{1}{2}$%, what will this sum amount to in the given time?

641. Given the time, rate, and amount, to find the principal and compound interest.

1. What sum of money will amount to \$$2902.263\frac{3}{8}$ in 20 years at 7% comp. int., and what will be the comp. int.?

OPERATION.

$$\begin{array}{ll} \$3.8696845 = \text{Amt. } \$1 \text{ for 20 yr. at } 7\%. & \\ \$2902.263\tfrac{3}{8} \div 3.8696845 = \$750, \text{ Prin.} & \\ \$2902.263 - \$750 = \$2152.26, \text{ Comp. Int.} & \end{array} \Big\} \textit{Ans.}$$

SOLUTION. — Consulting the table, we find that \$1 in 20 years at 7% will amount to \$3.8696845. Therefore \$$2902.263\frac{3}{8}$ must be the amount of \$$2902.263\frac{3}{8} \div 3.8696845$ which is \$750, and the compound interest is the difference between the amount and principal.

RULE. — I. *Divide the given amount by the amount of \$1 for the given time at the given rate; the result will be the principal.*

II. *Subtract the principal from the amount to find the compound interest.*

[redacted]

Find the principal and compound interest:

2. Amt. $ 8644.62; rate 6%; time 4 yr.

3. Amt. $ 5788.125; rate 5%; time 3 yr.

642. Given the principal, time, and compound interest, or amount, to find the rate.

1. If $ 87.4048 compound interest is paid for the use of a principal of $ 700 for 3 years, what is the rate per cent?

OPERATION.

$ 700 + $87.4048 = $ 787.4048, Amt. of $ 700 for 3 yr.

$ 787.4048 ÷ 700 = $ 1.124864, Amt. of $ 1.

By table the rate for 3 yr. = 4%. *Ans.*

SOLUTION.—Since $ 700 amounts to $ 787.4048, $ 1 will amount to $ 787.4048 ÷ 700 = $ 1.124864. Consulting the table in the line for 3 years, we find this amount under 4%.

RULE.—*Divide the amount by the principal; the result will be the amount of* $ 1 *for the given time. Find by the table the rate which for the given number of years will produce this amount.*

2. At what rate will $ 5000 amt. to $ 5304.50 in 2 yr.?

3. What is the rate if $ 10000 gains $ 7908.477 in 10 yr.?

643. Given the principal, rate, and compound interest, or amount, to find the time.

1. In what time will $ 500 gain $ 72.45, at 7% comp. int.?

OPERATION.

$ 500 + 72.45 = $ 572.45, Amt.

$ 572.45 ÷ 500 = $ 1.1449,

Amt. of $ 1 at 7%.

By table, time = 2 yr. *Ans.*

SOLUTION.—Since $ 500 amounts to $ 572.45 in a certain time, $ 1, in the same time at the same rate, will amount to $\frac{1}{500}$ of $ 572.45, which is $ 1.1449. By the table we find the time in which $ 1 will amount to $ 1.1449, at 7%, to be 2 yr.

RULE.—*Divide the amount by the principal; the result will be the amount of* $ 1 *for the given time. Find by the table the time which at the given rate will produce this amount.*

2. In what time will $ 1000 gain $1171.353 at 6%?

3. When will $ 333 amt. to $ 376.76 at 5%, semiannually?

4. *In what time will a sum double itself at 6% comp. int.?*

PARTIAL PAYMENTS OR INDORSEMENTS.

644. A **Partial Payment** is payment in part of a note, bond, or other obligation that is due or is drawing interest.

When the amount of a payment is written on the back of the obligation, it serves the purpose of a receipt, and is called an **Indorsement**.

Promissory Notes are written promises to pay certain sums of money on demand, or at specified times, as "30 days after date," "two months after date," etc. (§ 660). They constitute one of the most important forms of *commercial paper*. Other forms, as checks, drafts, etc., will be described under other heads.

For definitions of *maker*, *payee*, *indorser*, *negotiable note*, *face proceeds*, etc., see under "Bank Discount" pp. 384–386.

645. To secure uniformity in the method of computing interest when partial payments have been made, the Supreme Court of the United States has decided that:

"The rule for casting interest when partial payments have been made is to apply the payment, in the first place, to the discharge of the interest then due.

"If the payment *exceeds* the interest the surplus goes towards discharging the principal, and the subsequent interest is to be computed on the balance of the principal remaining due.

"If the payment is *less than* the interest the surplus of interest must not be taken to augment the principal, but the interest continues on the former principal until the period when the payments, taken together, exceed the interest due, and then the surplus is to be applied toward discharging the principal, and the interest is to be computed on the balance as aforesaid." — *Decision of Chancellor Kent.*

PRINCIPLES. — I. *Accrued interest must be paid before the principal can be diminished.*

II. *Interest must never draw interest.*

646. This decision has been adopted by nearly all the states of the Union.

[redacted]

UNITED STATES RULE. — I. *Find the amount of the given principal to the time of the first payment, and if this payment equals or exceeds the interest then due, subtract it from the amount obtained, and treat the remainder as a new principal.*

II. *But if the interest is greater than any payment, find the amount on the same principal to a time when the sum of the payments equals or exceeds the interest then due; subtracting the sum of the payments from that amount, the remainder will form a new principal, on which interest is to be computed as before.*

Examples.

647. To compute partial payments by the United States rule.

$2000. Springfield, Mass., Jan. 4, 1894.

1. *For value received I promise to pay James Parish, or order, two thousand dollars, one year after date, with interest. George Jones.*

Indorsements: Feb. 19, 1895, $400; June 29, 1896, $1000; Nov. 14, 1896, $520.

How much remained due Dec. 24, 1897?

OPERATION.

Principal on interest from Jan. 4, 1894	$2000
Interest to Feb. 19, 1 yr. 1 mo. 15 da.	135
Amount	$2135
Payment Feb. 19, 1895	400
Remainder for a new principal	$1735
Interest from Feb. 19, 1895, to June 29, 1896, 1 yr. 4 mo. 10 da.	141.69
Amount	$1876.69
Payment, June 29, 1896	1000
Remainder for a new principal	$876.69
Interest from June 29, 1896, to Nov. 14, 1896, 4 mo. 15 da.	19.725
Amount	$896.415
Payment, Nov. 14, 1896	520
Remainder for a new principal	$376.415
Interest from Nov. 14, 1896, to Dec. 24, 1897, 1 yr. 1 mo. 10 da.	25.09
Remainder due Dec. 24, 1897	Ans. $401.505

$ 475.50. ATLANTA, GA., April 1, 1893.

2. For value received, we jointly and severally promise to pay Mason & Brothers, or order, four hundred seventy-five dollars fifty cents, nine months after date, with interest at 7%.

JONES, SMITH & CO.

This note was indorsed as follows: Nov. 25, 1893, $ 50; June 10, 1894, $ 15.75; Aug. 1, 1895, $ 25.50; May 14, 1896, $ 104. How much was due March 15, 1897?

OPERATION.

Principal on interest from April 1, 1893		$ 475.50
Interest to Nov. 25, 1893, 7 mo. 24 da.		21.63
Amount		$ 497.13
Payment Nov. 25, 1893		50
Remainder for a new principal		$ 447.13
Interest from Nov. 25, 1893, to May 14, 1896, 2 yr. 5 mo. 19 da.		77.29
Amount		$ 524.42
Payment June 10, 1894, less than interest then due	$ 15.75	
Payment Aug. 1, 1895	25.50	
Their sum less than interest then due	$ 41.25	
Payment May 14, 1896	104	
Their sum exceeds the interest then due		$ 145.25
Remainder for a new principal		$ 379.17
Interest from May 14, 1896, to March 15, 1897, 10 mo. 1 da. .		22.19
Balance due March 15, 1897 . . . . *Ans.*		$ 401.36 +

$ 1000. BUFFALO, N. Y., May 15, 1888.

3. Two years after date I promise to pay to David Hudson, or order, one thousand dollars, with interest, for value received.

HENRY BURR.

Indorsements: Sept. 20, 1889, $ 150.60; Oct. 25, 1891, $ 200.90; July 11, 1893, $ 75.20; Sept. 20, 1894, $ 112.10; Dec. 5, 1895, $ 105. How much was due May 20, 1896?

$ 514.96. SAN FRANCISCO, June 20, 1892.

4. Three years after date we promise to pay Ross & Wade, or order, five hundred fourteen and $\frac{96}{100}$ dollars, for value received, with 10% interest. WILDER & BRO.

Indorsements: Nov. 12, 1892, $ 105.50; March 20, 1894, $ 200; July 10, 1894, $ 75.60. How much is due at maturity?

[redacted]

$3000. CHARLESTON, May 7, 1893.

5. For value received, I promise to pay George Babcock three thousand dollars, on demand, with 7% interest.

JOHN MAY.

Indorsements: Sept. 10, 1893, $25; Jan. 1, 1894, $500; Oct. 25, 1894, $75; April 4, 1895, $1500. How much was due Feb. 20, 1896?

$ $384\frac{95}{100}$. SAVANNAH, GA., Sept. 4, 1892.

6. Six months after date I promise to pay John Rogers, or order, three hundred eighty-four and $\frac{95}{100}$ dollars, for value received, with interest at 7%.

WILLIAM JENKINS.

This note was settled Jan. 1, 1894, one payment of $126.50 having been made Oct. 20, 1893. How much was due at the time of settlement?

$3475. NEW ORLEANS, March 6, 1889.

7. On demand we promise to pay Evans & Hart, or order, three thousand four hundred seventy-five dollars, for value received, with interest at 5%.

DAVIS & BROTHER.

Indorsements: June 1, 1889, $1247.60; Sept. 10, 1889, $1400. How much was due Jan. 31, 1890?

$500. PHILADELPHIA, Feb. 1, 1893.

8. For value received, I promise to pay J. B. Lippincott & Co., or order, five hundred dollars three months after date, with interest at 6%.

JAMES MONROE.

Indorsements: May 1, 1893, $40; Nov. 14, 1893, $8; April 1, 1894, $12; May 1, 1894, $30. How much was due Sept. 16, 1894?

648. The following method of computation is often used by merchants in the settlement of notes and of interest accounts *running a year or less:*

MERCANTILE RULE. — I. *Find the amount of the principal from the date of the note to the time of settlement.*

II. *Find the amount of each payment from the time it was made to the time of settlement.*

III. *Subtract the sum of the amounts of the payments from the amount of the principal, and the remainder will be the sum due.*

NOTE. — An accurate application of this rule requires that exact interest be computed by the rule for days. (See § 629.)

649. To compute partial payments by the Mercantile Rule.

1. On a note for $ 600 at 7%, dated Feb. 15, 1893, the following indorsements were made: March 25, 1893, $ 150; June 1, 1893, $ 75; Oct. 10, 1893, $ 100. How much was due Dec. 31, 1893?

OPERATION.

Am't of $ 600 from	Feb. 15 to Dec. 31,	319 da.,		$ 636.71	
" " $ 150 "	Mar. 25 " "	281 "	$ 158.08		
" " $ 75 "	June 1 " "	213 "	78.06		
" " $ 100 "	Oct. 10 " "	82 "	101.57	337.71	
	Balance due Dec. 31, 1893,			$ 299.00	*Ans.*

2. A note for $ 950, dated Jan. 25, 1892, payable in 9 mo., at 7% interest, had the following indorsements: March 2, 1892, $ 225; May 5, 1892, $ 174.19; June 29, 1892, $ 187.50; Aug. 1, 1892, $ 79.15. What was the balance due at the time of its maturity by the mercantile rule?

3. Payments were made on a debt of $ 1750, dated April 5, 1891, as follows: May 10, 1891, $ 190; July 1, 1891, $ 230; Aug. 5, 1891, $ 645; Oct. 1, 1891, $ 372. What was due Dec. 31, 1891, interest at 6%, by the mercantile rule?

$ 2000. BROOKLYN, N.Y., Sept. 1, 1895.

4. For value received, I promise to pay John Bartlett, or order, two thousand dollars, on demand, with interest annually at 6%. HAROLD WILLIAMS.

On this note were indorsed the following payments: Oct. 1, 1895, $ 350; Nov. 1, 1895, $ 225; Feb. 1, 1896, $ 500; March 1, 1896, $ 200; April 1, 1896, $ 50. How much was due on the note June 1, 1896, by the mercantile rule?

[redacted]

VERMONT AND NEW HAMPSHIRE RULES.

650. The following is the rule for partial payments in Vermont:

When the note or debt draws simple interest.

"On notes, bills, or other similar obligations, payable on demand or at a specified time, *with interest*, when payments are made, such payments shall be applied: first, to liquidate the interest accrued at the time of such payments; and second, to extinguish the principal." (See U. S. Rule, § 645.)

When the note or debt draws annual interest.

"When such obligations are payable on demand or at a specified time, *with interest annually*, the annual interest that remains unpaid shall bear simple interest from the time it becomes due to the time of final settlement; but if in any year, reckoning from the time such annual interest began to accrue, payments are made, the amount of such payments at the end of such year, with interest thereon from the time of payment, shall be applied: first, to liquidate the simple interest accrued from the unpaid annual interest; second, to liquidate the annual interests due; and, third, to extinguish the principal."

651. From this it will be seen that the United States Rule is followed *unless the note states with interest annually.* When the note specifies *with interest annually,* the following is the rule.

VERMONT RULE. — I. *Compute annual interest upon the principal to the end of the first year in which any payments are made.*

II. *Compute interest upon the payment or payments made in any one year from the time they were severally made to the end of the year in which they were made.*

III. *Apply the amount of such payment or payments: first, to discharge any interest that may have accrued upon the unpaid yearly interests; secondly, to discharge the yearly interests themselves; and, thirdly, toward extinguishing the principal.*

IV. *Proceed in a similar manner with succeeding payments.*

NOTE. — In Vermont a month of any number of days is considered as $\frac{1}{12}$ of a year, and a day as $\frac{1}{30}$ of a month; but in finding the time between two dates less than a month apart, *the actual number of days* must be counted.

652. The New Hampshire Rule is the same as that of Vermont with this additional provision:

NEW HAMPSHIRE RULE. — *If at the date of any payment, there is no interest due except what is accruing during the year, and the payment or payments do not exceed the interest due at the end of the year, deduct the payment or payments, without interest on the same.*

653. Although the United States Rule is now generally adopted in Connecticut, occasionally the old Connecticut Courts' Rule is still used, which is as follows:

CONNECTICUT RULE. — *Compute the interest to the time of the first payment; if that be one year or more from the time the interest commenced, add it to the principal, and deduct the payment from the sum total.*

If there be after payments made, compute the interest on the balance due to the time of the next payment, and then deduct the payment as above; and in like manner from one payment to another, till all the payments are absorbed, provided the time between one payment and another be one year or more.

But if any payment be made before one year's interest hath accrued, then compute the interest on the principal sum due on the obligation for one year, add it to the principal, and compute the interest on the sum paid, from the time it was paid, up to the end of the year; add it to the sum paid, and deduct that sum from the principal and interest added as above.

If any payments be made of a less sum than the interest arisen at the time of such payment, no interest is to be computed but only on the principal sum for any period.

654. To compute partial payments by the Vermont rule.

$2000. BURLINGTON, VT., May 10, 1890.

1. For value received, I promise to pay David Camp, or order, two thousand dollars, on demand, with 6% interest annually. RICHARD THOMAS.

On this note were indorsed the following payments: March 10, 1891, $800; May 10, 1893, $4; Sept. 10, 1894, $200; Nov. 10, 1894, $200. How much was due Jan. 10, 1895?

OPERATION.

	Payments.	Interest on Unpaid Yearly Interest.	Simple Interest.	Principal.
Principal, May 10, 1890				$ 2000
Simple interest on principal to May 10, 1891 (1 yr.)			$ 120	
Amount of first payment ($ 800 + int. for 2 mo.) .	$ 808			
Balance of first payment left to apply against principal after paying interest ($ 808 – $ 120) . .				688
Principal, May 10, 1891				$ 1312
Simple interest on principal to May 10, 1893 (2 yr.)			$ 157.44	
Interest on yearly interest of prin. ($78.72) for 1 yr.		$ 4.72		
Second payment	$ 4			
Balance of unpaid interest upon unpaid interest ($ 4.72 – $ 4.00)		$ .72		
Principal, May 10, 1893				$ 1312
Simple interest on principal to Jan. 10, 1895 (20 mo.)			$ 131.20	
Interest on yearly int. of prin. ($ 78.72) for 8 mo.		$ 3.15		
Balance of unpaid simple interest			157.44	
Interest on unpaid simple int. ($ 157.44) for 20 mo.		15.74		
Balance of interest on unpaid interest		.72		
Total interest on unpaid interest		$ 19.61		
Total unpaid simple interest			$ 288.64	
Amount of 3d payment ($ 200 + interest for 4 mo.)	$ 204			
Amount of 4th payment ($ 200 + interest for 2 mo.)	202			
Sum of payments	$ 406			
Balance to apply against principal after paying all interest ($ 406 – $ 19.61 – $ 288.64)				97.75
Amount due Jan. 10, 1895				$ 1214.25 *Ans.*

$ 600. Rutland, Vt., April 11, 1892.

2. For value received, I promise to pay Amos Cushing, or order, six hundred dollars on demand, with interest annually at 6%. John Brown.

Indorsements: Aug. 10, 1892, $ 156; Feb. 12, 1893, $ 200; June 1, 1894, $ 185. What was due Jan. 1, 1895?

$ 575. Keene, N. H., Aug. 4, 1894.

3. For value received, I promise to pay George Cooper, or order, five hundred seventy-five dollars on demand, with interest annually at 6%. David Green.

Indorsements: Nov. 4, '94, $ 64; Dec. 13, '95, $ 48; March 16, '96, $ 248; Sept. 28, '96, $ 60. What was due Nov. 4, '96?

TRUE DISCOUNT.

655. **True Discount** is an abatement or allowance made for the payment of a debt before it is due.

656. The **Present Worth** of a debt, payable at a future time without interest, is such a sum as, being put at legal interest, will amount to the given debt when it becomes due.

The **True Discount** is the difference between the whole debt and the present worth. It is the interest on the present worth of a debt from to-day to the time it matures.

657. In True Discount we observe the following relations:

I. The *Present Worth* corresponds to *Principal.*

II. The *Discount* corresponds to *Interest.*

III. The *Debt* corresponds to *Amount.*

Examples.

658. To find the present worth and true discount.

1. What is the present worth and what the true discount of $ 642.12 to be paid 4 yr. 9 mo. 27 da. hence, money being worth 7 % ?

OPERATION.

$ 1.33775, Amount of $ 1.

$ 642.12 ÷ 1.33775 = $ 480

$ 642.12, Given Sum.

$$\left.\begin{array}{ll} 480. & \text{Present Worth} \\ \hline \$\,162.12, & \text{True Discount} \end{array}\right\}\textit{Ans.}$$

SOLUTION. — Since $ 1 is the present worth of $ 1.33775 for the given time at the given rate of interest, the present worth of $ 642.12 must be as many dollars as 1.33775 is contained times in $ 642.12. Dividing, we obtain $ 480 for the present worth, and subtracting this sum from the given sum, we have $ 162.12, the true discount.

RULE. — I. *Divide the given sum or debt by the amount of* $ 1 *for the given rate and time; the quotient will be the present worth of the debt.*

II. *Subtract the present worth from the given sum or debt; the remainder will be the discount.*

NOTE. — When payments are to be made at different times without interest, find the *present worth of each* payment separately. Their sum will be the present worth of the *several payments*, and this sum subtracted from the sum of the several payments will leave *the total discount.*

[redacted]

2. What is the present worth of a debt of \$ $385.31\frac{1}{4}$, to be paid in 5 mo. 15 da., at 6% ?

3. How much should be discounted for the present payment of a note for \$ 429.986, due in 1 yr. 6 mo. 1 da., money being worth $5\frac{1}{2}$% ?

4. A man bought a farm for \$ 2964.12 ready money, and sold it again for \$ 3665.20, payable in 1 yr. 6 mo. How much would be gained in ready money, discounting at the rate of 8% ?

5. A man bought a mill with the option of paying \$ 25000 cash, or \$ 12000 payable in 6 mo. and \$ 15000 payable in 1 yr. 3 mo. He accepted the latter offer. Did he gain or lose, and how much, money being worth to him 10% ?

6. B bought a house and lot April 1, 1892, for which he was to pay \$ 1470 on the fourth day of the following September, and \$ 2816.80 Jan. 1, 1893. If he could get a discount of 10% for present payment, how much would he gain by borrowing the sum at 7%, and how much must he borrow ?

7. What is the difference between the interest and the true discount of \$ 576, due 1 yr. 4 mo. hence, at 6% ?

8. A merchant holds two notes against a customer — one for \$ 243.16, due May 6, 1893, and the other for \$ 178.64, due Sept. 25, 1893. How much ready money would cancel both the notes Oct. 11, 1892, discounting at the rate of 7% ?

9. A speculator bought a quantity of cotton for \$ 5270.40, on a credit of 9 months. He immediately sold the cotton for \$ 6441.60 cash, and paid the debt at 8% discount. How much did he gain ?

10. Which is the more advantageous, to buy flour at \$ 4.25 a barrel on 6 months, or at \$ 4.50 a barrel on 9 months, money being worth 8% ?

11. How much may be gained by borrowing money at 5% to pay off a debt of \$ 6400, due 8 months hence, allowing the present worth of this debt to be reckoned by deducting 5% *per annum* discount ?

BANK DISCOUNT.

659. A **Bank** is a corporation, chartered under the law, for the purpose of receiving and loaning money and issuing bank bills.

660. A **Promissory Note** is a written or printed obligation to pay a certain sum either on demand or at a specified time.

661. A **Check** is a written order on a bank by a depositor for money standing to his credit.

662. **Bank Notes**, or **Bank Bills**, are the notes made and issued by banks to circulate as money. They are payable in specie at the banks.

A bank which issues notes to circulate as money is called a **Bank of Issue**; one which lends money, a **Bank of Discount**; and one which takes charge of money belonging to other parties, a **Bank of Deposit**. Some banks perform two, and some all of these functions. National banks are organized under special legislation of Congress. They receive and lend money and issue notes which are received as money. Their circulating notes are secured by U.S. bonds deposited with the government.

663. The **Maker** of a note is the person by whom the note is signed. The **Payee** is the person to whose order the note is made payable. The **Holder** is the owner or his agent.

FORM OF PROMISSORY NOTE.

$500.00. *New York, July 1, 1895.*

Two months after date we promise to pay to the order of William Byrd

Five hundred $\frac{00}{100}$ Dollars

at The National Park Bank of New York

Value received. AMERICAN BOOK COMPANY,

By H. T. Ambrose,

No. 1357. Due Sept. 1. *Treasurer.*

FORM OF CHECK.

New York, Jan. 1, 1895. *No. 7056.*

THE NATIONAL PARK BANK OF NEW YORK.

Pay to the order of ~~~~ *James Ball* ~~~~ *$50.75*

~~~~ *Fifty* ~~~~ $\frac{75}{100}$ *Dollars.*

AMERICAN BOOK COMPANY,

By H. T. Ambrose,

Treasurer.

664. A **Negotiable Note** is one which may be bought and sold, or negotiated. It is made payable to *the bearer* or to *the order* of the payee.

665. **Indorsing** a note by a payee or holder is the act of writing his name on its back.

NOTES. — 1. If a note is payable to the bearer, it may be negotiated without indorsement.

2. An indorsement makes the indorser liable for the payment of a note, if the maker fails to pay it when it is due.

3. A note should contain the words "value received," and the sum for which it is given should be written out in words.

666. The **Face** of a note is the sum specified in the note.

667. **Days of Grace** are the three days usually allowed by law for the payment of a note after the expiration of the time specified in the note.

668. The **Maturity** of a note is the expiration of the term specified for its payment including the days of grace, where they are allowed.

NOTES. — 1. No grace is allowed on notes payable "on demand," without grace. In New York, Vermont, California, and some other states, days of grace have been recently abolished by statute. The maturity of such notes is the expiration of the time mentioned in them.

2. To indicate the maturity of a note or draft, an oblique line is used, with the day at which the note is nominally due on the left, and the date of maturity on the right; thus, *Jan.* $^7/_{10}$.
~~~~

669. The **Term of Discount** is the time from the date of discount of a note to its maturity. Notes may contain a promise of interest, which will be reckoned from the date of the note, unless some other time is specified.

NOTE. — A note not paid at maturity draws interest from the day it is due, even though no mention is made of interest in the note.

670. A **Notary,** or **Notary Public,** is an officer authorized by law to attest documents or writings to make them authentic.

671. A **Protest** is a formal declaration in writing, made by a Notary Public, at the request of the holder of a note, notifying the maker and the indorsers of its non-payment.

NOTE. — The failure to protest a note on the third day of grace releases the indorsers from all obligation to pay it, except when the indorser has written above his signature the words "*protest waived.*" An indorser who writes "*without recourse*" above his signature is not liable for payment under any circumstances. If the third day of grace or the maturity of a note occurs on Sunday or a legal holiday, in most places the note must be paid on the day previous; but the time of protest for non-payment varies by laws of different places.

672. **Bank Discount** is the interest charged by a bank for the payment of a note before it becomes due.

673. The **Proceeds** of a note is the sum received for it when discounted, and is equal to the face of the note less the discount.

The transaction of borrowing money at banks is conducted in accordance with the following custom: The borrower presents a note, either made or indorsed by himself, payable at a specified time, and receives for it a sum equal to the face *less* the interest for the time the note has to run. The amount thus withheld by the bank is in consideration of advancing money on the note prior to its maturity.

674. The law of custom at banks makes the bank discount of a note equal to the simple interest at the legal rate, for the time specified in the note. As the bank always takes the interest at the time of discounting a note, bank discount is equal to simple interest *paid in advance* upon the sum due on a note at its maturity, or it is equal to simple interest *plus the interest on the interest* for the term of discount.

Thus, the true discount of a note for \$153, which matures in 4 mo. at 6%, is $\$153 - \frac{15300}{102} = \$3.00$, and the bank discount is $\$153 \times .02 = \$3.06$. Since the interest of \$3, the true discount, for 4 mo. is $\$3 \times .02 = \$.06$, *we see that the bank discount of any sum for a given time exceeds the true discount, by the interest on the true discount for the same time.*

675. Bank discount is computed on the *face* of a note that does not bear interest, but on the *face plus the interest* of a note that bears interest.

676. In computations in bank discount, the *Face* (or *Face plus interest*) of the note corresponds to the *Principal*, the *Bank Discount* to the *Interest*, the *Proceeds* to the *Difference*, or *Principal minus interest*; and *Rate of Discount* to the *Rate of Interest.*

Examples.

677. Given the face of a note, to find the bank discount and the proceeds.

1. What are the proceeds and bank discount of a note for \$2000, due in 2 mo. 15 da. at 6% ?

OPERATION.

$$\text{Time} = 2 \text{ mo. } 18 \text{ da.}$$
$$\$.013 = \text{Int. of } \$1 \text{ for } 2 \text{ mo. } 18 \text{ da.}$$
$$\left.\begin{array}{l} \$.013 \times 2000 = \$26, \text{ Bank Discount} \\ \$2000 - \$26 = \$1974, \text{ Proceeds} \end{array}\right\} \textit{Ans.}$$

SOLUTION. — The time + 3 da. = 2 mo. 18 da. The int. of \$1 for 2 mo. 18 da. = \$.013. The int. of \$2000 = 2000 × \$.013 = \$26, bank disc. The face of the note, \$2000 – the bank disc., \$26, = the proceeds, \$1974.

RULE. — I. *Compute the interest on the face of the note for three days more than the specified time; the result will be the discount.*

II. *Subtract the discount from the face of the note, and the remainder will be the proceeds.*

2. What is the bank discount, and what are the proceeds of a note for \$1487 due in 30 days at 6%?

3. What are the proceeds of a note for \$384.50 at 90 days, if discounted at 6% ?

4. Wishing to borrow \$1000 of a Southern bank that is discounting paper at 8%, I give my note for \$975, payable in 60 days. How much more will make up the required *amount?*

Find the day of maturity, the term of discount, and the proceeds of the following notes:

$\$1962\frac{45}{100}$. DETROIT, July 26, 1892.

5. Four months after date I promise to pay to the order of James Gillis one thousand nine hundred sixty-two and $\frac{45}{100}$ dollars at the Exchange Bank, for value received.

JOHN DEMAREST.

Discounted Aug. 26, at 6%.

$\$1066\frac{75}{100}$. BALTIMORE, April 19, 1891.

6. Ninety days after date we promise to pay to the order of King & Dodge one thousand sixty-six and $\frac{75}{100}$ dollars at the Citizens' Bank, for value received.

CASE & SONS.

Discounted May 8, at 6%.

$\$784\frac{72}{100}$. MOBILE, June 20, 1893.

7. Two months after date for value received I promise to pay George Thatcher or order seven hundred eighty-four and $\frac{72}{100}$ dollars at the Traders' Bank.

WILLIAM HAMILTON.

Discounted July 5, at 8%.

8. What is the difference between the true and the bank discount of $950, for 3 months at 7%?

678. Given the proceeds of a note, to find the face.

1. For what sum must I draw my note at 4 months, interest 6%, that the proceeds when discounted in bank shall be $750?

OPERATION.

$1.0000
 .0205, Disc't on $1 for 4 mo. 3 da.

$ .9795, Proceeds of $1.

$\$750 \div .9795 = \$765.696$ *Ans.*

SOLUTION — We first obtain the proceeds of $1; then, since $.9795 is the proceeds of $1, $750 is the proceeds of as many dollars as .9795 is contained times in $750. Dividing, we obtain the required result.

RULE. — *Divide the proceeds by the proceeds of* $1 *for the time and rate mentioned;* the quotient will be the face of the note.

[redacted]

2. What is the face of a note at 60 days, the proceeds of which, when discounted at bank at 6%, are \$ 1275?

3. If a merchant wishes to draw \$ 5000 at bank, for what sum must he give his note at 90 days, discounting at 6%?

4. The avails of a note having 3 months to run, discounted at a bank at 7%, were \$ 276.84. What was the face of the note?

5. James T. Fisher buys a bill of merchandise at cash price, to the amount of \$ 1486.90, and gives in payment his note at 4 months at $7\frac{1}{2}$%. What must be the face of the note?

6. Find the face of a 6 months' note, the proceeds of which, discounted at 2% a month, are \$ 496.

7. For what sum must a note be drawn at 30 days, to net \$ 1200 when discounted at 5%?

8. Owing a man \$ 575, I give him a 60 day note. What should be the face of the note, to pay him the exact debt, if discounted at $1\frac{1}{2}$% a month?

679. Given the rate of bank discount, to find the corresponding rate of interest.

1. A broker discounts 30 day notes at 8%. What rate of interest does his money earn him?

OPERATION.

30 day notes = 33 days' time.

\$ 100.00 Base.

\$ $.73\frac{1}{3}$, Discount for 33 days.

\$ $99.26\frac{2}{3}$, Proceeds.

Interest of \$ $99.26\frac{2}{3}$ for 33 da. at 1% = \$ $.0909\frac{17}{18}$.

\$ $.73\frac{1}{3}$ ÷ \$ $.0909\frac{17}{18} = 8\frac{968}{16379}$% *Ans.*

SOLUTION. — If we assume \$ 100 as the face of the note, the discount for 33 days at 8% will be \$ $.73\frac{1}{3}$ and the proceeds \$ $99.26\frac{2}{3}$. We then have \$ $99.26\frac{2}{3}$ principal, \$ $.73\frac{1}{3}$ interest, and 33 days' time, to find the rate per cent per annum, which we do by § 632.

RULE. — I. *Find the discount and the proceeds of* \$ 1 *or* \$ 100 *for the time the note has to run.*

II. *Divide the discount by the interest of the proceeds at* 1% *for the same time.*

2. What rate of interest is paid, when a note payable in 30 days is discounted at 6% ?

3. A note payable in 2 months is discounted at 2% a month. What rate of interest is paid ?

4. When a note payable in 90 days is discounted at 1% a month, what rate of interest is paid ?

5. What rate of interest corresponds to a 5, 6, 7, 10, 12% discount on a note running 10 months without grace ?

680. Given the rate of interest, to find the corresponding rate of bank discount.

1. A broker buys 60 day notes at such a discount that his money earns him 8% a year. What is his rate % of discount?

OPERATION.

60 da. + 3 da. = 63 da.

$100, Base.
1.40, Int. for 63 da.
$101.40, Amount " "

Int. of $101.40 for 63 da., at 1% = $.17745.

$\$1.40 \div \$.17745 = 7\frac{451}{507}\%$ *Ans.*

SOLUTION.—If we assume $100 as the proceeds of a note, the interest for 63 days at 8% will be $1.40, and the amount or face of the note will be $101.40. We then have $101.40 the principal, $1.40 the interest, and 63 days the time, to find the rate per cent, which we do by § 632, as in the last case.

RULE.—I. *Find the interest and the amount of $1 or $100 for the time the note has to run.*

II. *Divide the interest by the interest on the amount at 1% for the same time.*

2. What rates of bank discount on 30 day notes correspond to a 5, 6, 7, 10% interest ?

3. At what rate should a 3 months' note be discounted to produce 8% interest ?

4. At what rate must a note payable 18 months hence, *without grace*, be discounted to produce 7% interest ?

[redacted]

SAVINGS BANK ACCOUNTS.

681. Savings Banks are institutions intended to receive in trust or on deposit small sums of money, generally the surplus earnings of laborers, and to return the same with a moderate interest at a future time.

682. Money deposited on or before certain fixed dates draws interest from those dates. In some banks the interest term is 6 months, in some 3 months, and in others 1 month, each beginning with the *first day* of the month.

If the interest is not drawn when due, it is added to the principal and draws interest as a new deposit; hence we see that savings banks pay *compound interest*.

683. A savings bank furnishes each depositor with a book, in which are recorded from time to time the sums deposited and the sums drawn out. The Dr. side of such an account shows the deposits, and the Cr. side the depositor's checks or drafts.

Savings banks usually allow interest only on such sums as have been on deposit for a *full interest term*. Thus, if interest is paid semiannually and the times of payment are Jan. 1 and July 1, interest will be allowed only on money deposited on or before Jan. 1 and not withdrawn before July 1, or deposited before July 1 and not withdrawn before Jan. 1. If interest is payable quarterly, the sum must be on deposit for three full months from the date of interest payment, if payable monthly for 1 month, etc. The interest term of most banks is 3 months.

NOTES. — 1. The smallest balance on deposit any time during the term is considered the smallest balance on deposit during the whole term.

2. An exception to the general rule occurs in the practice of some of the savings banks of New York City. In these, the interest term is 6 months, and the depositor is allowed not only the full term's interest on the smallest balance, but a half term's interest on any deposit, or portion of a deposit made during the first 3 months of the term, and *not drawn out during any subsequent part of the term*.

3. In some banks money deposited previous to the 1st day of any month draws interest from that date to the day of declaring interest dividends, provided it has not been previously withdrawn.

4. No interest is allowed on fractional parts of a dollar.

5. Usually a depositor may withdraw any portion of his money whenever he chooses, but some banks require a week's notice.

PRINCIPLE. — *Interest is added at the end of each interest term on the smallest balance on deposit during the entire term.*

Examples.

684. To find the interest or balances due from savings banks.

1. Find the balance due July 1, 1895, on the following account, interest being allowed quarterly at 4% per annum.

Dates.	Deposits.		Sums drawn out.		Interest.		Daily Balances.	
1894.								
Dec. 31	$150	00					150	00
1895.								
Jan. 15			50	00			100	00
Feb. 8	85	00					185	00
Feb. 27			15	00			120	00
Mar. 29	20	50					140	50
April 1					1	00	141	50
April 10	43	00					184	50
April 19			45	00			139	50
May 15	26	25					165	75
July 1	50	00			1	39	217	14

SOLUTION.—The interest is added to the daily balance, each quarter on the smallest balance on deposit during the quarter. The smallest balance on deposit during the first quarter is $100, and the interest due on this on April 1 (4% for 1 yr. = 1% for 3 months) is $1. Adding this interest to $140, the balance for April 1 is $141. The smallest balance during the second quarter is $139.50, and as no interest is allowed on a fractional part of a dollar, the interest on $139 is $1.39. Adding this interest and the deposit for July 1, the balance on hand July 1 is $217.14.

RULE.—*Add to the balance on hand at the beginning of each interest term the interest on the smallest balance on hand during the preceding term.*

2. What will be due April 20, 1896, on the following account, interest being allowed quarterly at 4% per annum, the terms commencing Jan. 1, April 1, July 1, and Oct. 1?

Dr. SAVINGS BANK, *in account with* JAMES TAYLOR. *Cr.*

1895.			1895.		
Jan. 12		$75	March 5		$30
May 10		150	Aug. 16		50
Sept. 1		20	Dec. 1		48
1896.					
Feb. 16		180			

[redacted]

NOTE. — In the following examples the terms commence with the year, or on Jan. 1.

3. Allowing interest monthly at 4% per annum, what sum will be due Sept. 1, 1896, on the book of a bank having the following entries?

Dr. BAY STATE SAVINGS INSTITUTION, *in account with* JANE LADD. *Cr.*

1896.				1896.			
Jan. 3	To Cash,	5	75	Jan. 28	By Check,	5	00
" 8	" "	13	45	Feb. 7	" "	8	48
" 20	" "	7	60	March 20	" "	10	00
Feb. 20	" Check,	16	45	April 11	" "	12	76
" 27	" Cash,	8	40	June 3	" "	3	96
March 6	" Check,	14	65	" 12	" "	10	48
" 29	" Cash,	7	98	" 20	" Draft,	17	48
April 25	" "	8	49	Aug. 17	" Check,	5	64
May 7	" Draft,	26	50				
" 30	" "	45	79				
July 28	" Cash,	15	68				
Aug. 8	" Check,	18	45				
" 26	" Cash,	4	50				

4. Interest being allowed quarterly, at 4%, how much was due April 4, 1896, on the following savings bank account?

Dr. DETROIT SAVINGS INSTITUTION, *in account with* R. L. SELDEN. *Cr.*

1895.				1895.			
Feb. 1	To Cash,	47	50	May 12	By Check,	50	86
March 12	" "	124	86	Oct. 3	" "	25	78
June 20	" "	130	56	Nov. 16	" "	36	48
Aug. 8	" "	68	75	Dec. 28	" "	12	50
1896.							
Jan. 25	" "	160	80				

5. How much was due Dec. 10, 1895, on the following account, allowing interest semiannually, at 4% per annum?

Dr. IRVINGS SAVINGS INSTITUTION, *in account with* JAMES TAYLOR. *Cr.*

1894.				1894.			
June 4	To Cash,	175		Sept. 14	By Check,	65	
Nov. 1	" "	150		1895.			
1895.				July 25	" "	120	
Feb. 24	" Draft,	200		Dec. 3	" "	80	
Sept. 10	" Check,	56					

6. With interest quarterly at 4%, how much was due July 1, 1895, on a book having the following entries?

Dr. SIXPENNY SAVINGS BANK, *in account with* WILLIAM GALLUP. *Cr.*

1894.				1894.			
April 1	To Check,	36	50	Sept. 16	By Check,	36	16
June 17	" "	25	38	1895.			
Nov. 1	" Cash,	84	72	Jan. 27	" "	18	48
				March 1	" "	17	50

EXCHANGE.

685. If a man wishes to make a remittance to a creditor, agent, or any other person residing at a distance, instead of transporting specie, which is attended with expense and risk, or sending bank notes, which are liable to be uncurrent at a distance from the banks that issue them, he remits a *bill of exchange*, purchased at a bank or elsewhere, and made payable to the proper persor in or near the place where he resides.

Thus a man by paying Boston funds in Boston, may put New York funds into the hands of his New York agent.

686. **Exchange** is a method of remitting money from one place to another, or of making payments by written orders, without transmitting money.

687. A **Bill of Exchange** is a written request or order upon one person to pay a certain sum to another person, or to his order, at a specified time.

688. A **Sight Draft** or **Bill** is one requiring payment to be made "at sight," which means, at the time of its presentation to the person ordered to pay. In **Time Drafts** the time specified is usually a certain number of days "after sight."

SIGHT DRAFT.

NO PROTEST.
Please take this off before presenting. If not paid at maturity, return at once, stating reasons.

AMERICAN BOOK COMPANY.

$200. New York, Sept. 1, 1895.

AT SIGHT pay to the order of Ourselves

Two hundred $\frac{00}{100}$ Dollars

Value received and charge to acct. of

To A. C. Dolan,
B'klyn, N.Y.

AMERICAN BOOK COMPANY,
By H. T. Ambrose,
Treasurer.

NOTE. — Other drafts have the same form as the above, except that instead of the words "at sight," "——— days after sight" or "——— days after date" are used. When the *time is after sight*, it means after acceptance.

[redacted]

689. There are always three parties to a transaction in exchange, and usually four:

The **Drawer** or **Maker** is the person who signs the order or bill.

The **Drawee** is the person to whom the order is addressed.

The **Payee** is the person to whom the money is ordered to be paid.

The **Buyer** or **Remitter** is the person who purchases the bill. He may be himself the *payee*, or the bill may be drawn in favor of any other person.

The **Indorsement** of a bill is the *writing* upon its back, by which the *payee* transfers the payment to another. The payee may indorse *in blank* by writing his name only, which makes the bill transferable like a bank note; or he may accompany his signature by a special order to pay to another person, who in his turn may transfer the title in like manner. *Indorsers* become separately liable for the amount of the bill, in case the *drawee* fails to make payment, but each indorser has recourse upon the drawer or maker of the note. A bill made payable to the *bearer* is transferable without indorsement.

NOTE. — On the back of the draft on p. 394 there might be one or more indorsements, as follows: "Pay to the order of the National Park Bank, for collection for account of American Book Co."; "Pay to the order of the First National Bank, B'klyn, for collection for account of National Park Bank, N.Y."; "First National Bank, B'klyn, N.Y. Paid."

690. The **Acceptance** of a bill is the promise which the *drawee* makes when the bill is presented to him to pay it at maturity; this obligation is usually acknowledged by writing the word "Accepted," with his signature and date across the face of the bill. The draft is then called an *acceptance* and may be negotiated like a promissory note.

NOTES. — 1. In this country and in Great Britain, three *days of grace* are allowed for the payment of a bill of exchange, after the time specified has expired. In regard to grace on *sight drafts*, however, custom is variable; in New York, and some other states, no grace is allowed on sight bills.

2. When a bill is protested for non-acceptance, the drawer is liable to pay it immediately, even though the specified time has not expired.

691. The **Face** of a bill of exchange is the sum ordered to be paid; it is usually expressed in the currency of the place on which the draft is made.

692. The **Rate of Exchange** is the current price paid in one place for bills of exchange on another place. This price varies, according to the relative conditions of trade and commercial credit at the two places between which exchange is made. Thus, if Boston is largely indebted to Paris, bills of exchange on Paris will bear a high price in Boston, and will

be at a premium, whereas bills of exchange on Boston will bear a low price in Paris, or be at a discount. Such premiums or discounts are reckoned at a certain per cent on the face of the bill.

When the rate of exchange between two places is unfavorable to drawing or remitting, the disadvantage is sometimes avoided, by means of a circuitous exchange on intermediate places between which the rate is advantageous.

693. **Direct Exchange** is confined to the two places between which the money is to be remitted.

694. There are always two methods of transmitting money between two places. Thus, if A is to receive money from B,

1st. A may draw on B, and sell the draft;

2d. B may remit a draft, made in favor of A.

NOTE. — One person is said to *draw on* another person, when he is the *maker* of a draft addressed to that person.

695. In computations in Exchange the *Face* of a draft corresponds to the *Base*, the *Premium* or *Discount* to the *Percentage*, the *Cost* of the draft to the *Amount* or *Difference*.

Exchange is of two kinds, — Domestic and Foreign.

DOMESTIC EXCHANGE.

696. **Domestic** or **Inland Exchange** relates to remittances made between different places of the same country.

NOTE. — An inland bill of exchange is commonly called a *draft*.

697. The rate of exchange for inland bills, or drafts, is always expressed by the rate of premium or discount. Drafts on time, however, are subject to *bank discount*, like promissory notes, for the term of credit given. Hence, their cost is affected by both the *rate of exchange* and the *discount for time*.

PRINCIPLES. — I. *The rate of exchange on a sight draft is* 100% *plus the per cent of premium or minus the per cent of discount.*

II. *The rate of exchange on a time draft is the rate on a sight draft less the per cent of* bank discount.

[redacted]

Examples.

698. To compute domestic exchange.

$ 500. SYRACUSE, May 7, 1896.

1. At sight, pay to James Clark, or order, five hundred dollars, value received, and charge the same to our account.

M. SMITH & Co.

To MESSRS. BROWN & FOSTER, Baltimore, Md.

What is the cost of the above draft, the rate of exchange being $1\frac{1}{2}\%$ premium?

OPERATION.

$\$ 500 \times 1.015 = \$ 507.50$ *Ans.*

SOLUTION.—Since exchange is at $1\frac{1}{2}\%$ premium, each dollar of the draft will cost \$ 1.015; and to find the whole cost of the draft, we multiply its face, \$ 500, by 1.015, and obtain \$ 507.50, the required result.

$ 600. PITTSBURG, June 12, 1896.

2. Sixty days after sight, pay to William Barnard, or order, six hundred dollars, value received, and charge the same to our account.

THOMAS BAUER & Co.

To the Suffolk Bank, Boston.

What will be the cost of the above draft, exchange on Boston being in Pittsburg at $2\frac{1}{4}\%$ premium?

OPERATION.

$\$ 1 + \$ .0225 = \$ 1.0225$, Rate of Exchange.
$.0105$, Bank Discount of \$ 1 (63 da.).
$\$ 1.012$, Cost of Exchange for \$ 1.

$\$ 600 \times 1.012 = \$ 607.20$ *Ans.*

SOLUTION.—From \$ 1.0225, the rate of exchange, we subtract \$ .0105, the bank discount of \$ 1 for the specified time, and obtain \$ 1.012, the cost of exchange for \$ 1; then $\$ 600 \times 1.012 = \$ 607.20$, the cost of exchange for \$ 600.

3. A commission merchant in Detroit wishes to remit to his employer in St. Louis \$ 512.36 by draft at 60 days. What is the face of the draft which he can purchase with this sum, *exchange being* at $2\frac{1}{2}\%$ discount?

OPERATION.

$$\$1 - \$.025 = \$.975, \text{ Rate of Exchange.}$$
$$\underline{.0105}, \text{ Discount of } \$1.$$
$$\$.9645, \text{ Cost of Exchange for } \$1.$$
$$\$512.36 \div .9645 = \$531.218 \quad \textit{Ans.}$$

Solution. — From \$.975, the rate of exchange, we subtract \$.0105, the bank discount of \$1 for the specified time, at the legal rate in Detroit, 6%, and obtain \$.9645, the cost of exchange for \$1; and the face of the draft that will cost \$512.36, will be as many dollars as .9645 is contained times in \$512.36, which is \$531.218.

Rule. — I. To find the cost of a draft, the face being given. — *Multiply the face of the draft by the cost of exchange for* \$1.

II. To find the face of a draft, the cost being given. — *Divide the given cost by the cost of exchange for* \$1.

4. What must be paid in New York for a draft on Boston, at 30 days, for \$5400, exchange being at $\frac{1}{2}$% premium?

5. What is the cost of sight exchange on New Orleans, for \$3000, at $3\frac{1}{4}$% discount?

6. What must be paid in Philadelphia for a draft on St. Paul drawn at 90 days, for \$4800, the rate of exchange being $101\frac{3}{8}$%?

7. A sight draft was purchased for \$550.62, exchange being at a premium of $3\frac{1}{2}$%. What was the face?

8. An agent in Syracuse, N.Y., having \$1324.74 due his employer, is instructed to remit the same by a draft drawn at 30 days. What will be the face of the draft, exchange being at $1\frac{3}{4}$% premium?

9. My agent in Charleston, S.C., sells a house and lot for \$7500, on commission of $1\frac{1}{2}$%, and remits to me the proceeds in a draft purchased at $\frac{1}{2}$% premium. What sum do I receive?

10. A man in Hartford, Conn., has \$4800 due him in Baltimore. How much more will he realize by making a draft for this sum on Baltimore and selling it at $\frac{1}{2}$% discount, than by having a draft on Hartford remitted to him, purchased in Baltimore for this sum at $\frac{3}{4}$% premium?

[redacted]

11. The Merchants' Bank of New York having declared a dividend of $6\frac{1}{4}\%$, a stockholder in Cincinnati drew on the bank for the sum due him, and sold the draft at a premium of $1\frac{3}{4}\%$, thus realizing $508.75 from his dividend. How many shares did he own?

Foreign Exchange.

699. **Foreign Exchange** relates to remittances made between different countries.

The drafts or bills of exchange are expressed in the money of the country in which they are made payable.

700. A **Set of Exchange** is a bill drawn in triplicate, named First, Second, and Third of exchange, each copy being valid until the amount of the bill is paid. These copies are sent by different mails, to provide against miscarriage. When one is paid, the others are *void*.

SET OF EXCHANGE.

£700. New York, Feb. 1, 1896.

At sight of this First of Exchange (Second and Third of the same tenor and date unpaid), pay to the order of Samuel Monmouth, *Seven Hundred Pounds Sterling*, for value received, and charge the same to the account of H. B. Claflin Co.

To Robins, Barclay & Co.,
London, England.

This is the form of the *first bill*; the *second* requires only the change of "First" into "Second," and of "Second and Third of the same tenor," etc., into "First and Third"; the *third* bill varies similarly.

701. The **Par of Exchange** is the estimated value of the coins of one country as compared with those of another, and is either *intrinsic* or *commercial*. (See pp. 204 and 205.)

The **Intrinsic Par of Exchange** is the comparative value of the coins of different countries, as determined by their weight and purity.

The **Commercial Par of Exchange** is the comparative value of the coins of different countries, as determined by their nominal or market price.

Note. — The intrinsic par is always the same while the coins remain unchanged; but the *commercial par, being* determined by commercial usage, is fluctuating.

702. Sterling Bills, or **Sterling Exchange**, are bills on England, Ireland, or Scotland, or any of the British colonies. Such bills are negotiated at a rate fixed without reference to the par of exchange.

703. Exchanges with Europe are effected chiefly through the following prominent financial circles: London, Paris, Antwerp, Amsterdam, Hamburg, Frankfort, Bremen, Berlin, Geneva, etc.

The rate of exchange varies with the amount of business and the direction of money remittances. In computing exchange, it is the custom in some cases to state the value of the United States monetary unit in units and fractions of a unit of foreign currency, and in other cases to express the value of the foreign monetary unit in dollars and fractions of a dollar United States money.

Thus, quotations of exchange on London give the value of *£ 1 sterling* in *dollars* and *cents*; on Paris, Antwerp, and Geneva, the value of *$ 1* in *francs*; on Hamburg, Frankfort, and Berlin, the value of *4 reichsmarks*, or *marks* in *cents*; on Amsterdam, the value of *1 guilder* or *florin* in *cents*.

NOTE. — Foreign bills of exchange are usually drawn "at sight" or "60 days after sight." The former are known as short bills and the latter as long bills.

704. The following table shows the manner in which quotations of foreign exchange are made in this country.

	LONG BILLS.	SIGHT DRAFTS.
Sterling	$4.88\frac{1}{4}$	$4.89\frac{1}{2}$
Francs	$5.16\frac{7}{8}$	$5.15\frac{5}{8}$
Reichsmarks	$95\frac{7}{16}$–$95\frac{1}{2}$	$95\frac{9}{16}$
Guilders	$40\frac{1}{4}$–$40\frac{5}{16}$	$40\frac{7}{16}$

Examples.

705. To find the cost of a foreign bill of exchange.

1. Find the cost in New York of the following bill on London, at 3 da. sight, exchange quoted at $4.87\frac{1}{2}$:

£ 500. NEW YORK, April 1, 1896.

At sight of this FIRST of Exchange (Second and Third unpaid), pay to the order of John Walker & Co., Five Hundred Pounds sterling, value received, and charge the same to the account of

BROWN BROTHERS & Co.

To BROWN, SHIPLEY & Co.,
London, England.

[redacted]

OPERATION.

$$\$4.875 \times 500 = \$2437.50 \quad \textit{Ans.}$$

SOLUTION. — Since £1 costs $4.875, £500 will cost 500 times $4.875 which is $2437.50.

2. Find the cost of a bill on Paris for 495 francs, at 5.15.

OPERATION.

$$495 \div 5.15 = \$96.12 \quad \textit{Ans.}$$

SOLUTION. — Since 5.15 francs cost $1, 495 francs will cost as many dollars as 5.15 francs are contained times in 495 francs, or $96.12.

3. Find the cost of a bill of exchange on Berlin for 1750 reichsmarks, quoted at $96\frac{1}{4}$.

OPERATION.

$$\$.9625 \div 4 \times 1750 = \$421.09 \quad \textit{Ans.}$$

SOLUTION. — Since $.9625 is the value of 4 reichsmarks, the value of 1 reichsmark is $\frac{1}{4}$ of $.9625, and the value of 1750 reichsmarks is 1750 times the quotient, which is $421.09.

4. Find the cost of a bill of exchange on Amsterdam, for 2000 guilders, quoted at $40\frac{1}{4}$.

OPERATION.

$$\$.40\tfrac{1}{4} \times 2000 = \$805 \quad \textit{Ans.}$$

SOLUTION. — Since 1 guilder costs $\$.40\frac{1}{4}$, 2000 guilders will cost 2000 times $\$.40\frac{1}{4}$ which is $805.

RULE. — *Multiply the face of the bill of exchange by the value of the foreign monetary unit in United States money.* Or,

Divide the face of the bill of exchange by the value of $1 *in the foreign monetary unit expressed decimally.*

5. Find the cost of a bill on Liverpool for £600 15*s.*, at $\$4.86\frac{1}{2}$.

6. Find the cost of a bill on Geneva, Switzerland, for 5460 francs, at $5.21\frac{1}{4}$.

7. Find the cost of a bill on Hamburg for 2560 reichsmarks, at $95\frac{5}{8}$.

8. *Find* the cost of a bill on Paris for 2400 francs at 5.16?

9. What is the cost in Portland of a bill on Manchester, England, for £ 325 3*s.* 9*d.*, when sterling exchange is selling at 4.89½?

10. What must be paid in Charleston for a bill of exchange on Paris for 6000 francs, exchange being at 5.31?

11. What is the cost in Boston of a bill on St. Petersburg for 3000 rubles, at $ .772; brokerage ¼%?

12. What will be the cost in Boston of the following bill of exchange on Liverpool, exchange being at 4.87½?

£ 432. BOSTON, June 16, 1893.

At sight of this FIRST of Exchange (Second and Third of same tenor and date unpaid), pay to the order of J. Simmons, Boston, Four Hundred Thirty-two Pounds, value received, and charge the same to account of JAMES LOWELL & Co.

TO RICHARD EVANS & SON,
Liverpool, England.

13. A merchant in Cincinnati has 9087½ guilders due him in Amsterdam, and requests the remittance by draft. What sum will he receive, exchange on U.S. in Amsterdam selling at 2.41 guilders for $ 1?

706. To find the face of a bill of exchange.

1. What will be the face of a bill on London, that can be bought in New York for $ 5488.26, exchange being quoted at 4.85?

OPERATION.

$\$5488.26 \div \$4.85 = 1131.6$, or £ 1131.6 = £ 1131 12*s.* *Ans.*

SOLUTION. — Since £ 1 = $ 4.85, $ 5488.26 will equal as many pounds as $ 4.85 is contained times in $ 5488.26, which is 1131.6 times, or £ 1131.6 = £ 1131 12*s.*

2. What will be the face of a bill on Paris, bought for $ 325, exchange at 5.17?

OPERATION.

5.17 francs × 325 = *1680.25* francs *Ans.*

SOLUTION. — Since $ 1 = 5.17 francs, $ 325 = 325 × 5.17 francs, = 1680.25 francs.

[redacted]

3. What will be the face of a bill on Hamburg, bought for $4000, exchange being quoted at 96?

OPERATION.

$$\$.96 \div 4 = \$.24.$$
$$\$4000 \div \$.24 = 16666\tfrac{2}{3} \text{ reichsmarks} \quad \textit{Ans.}$$

SOLUTION. — Since 4 reichsmarks = $.96, 1 reichsmark = ¼ of $.96 = $.24, and $4000 = as many reichsmarks as $.24 is contained times in $4000 = 16666⅔ times, or 16666⅔ reichsmarks.

RULE. — *Divide the cost of exchange by the value of the foreign monetary unit in United States money.* Or,

Multiply the cost of exchange by the value of $1 in the foreign monetary unit expressed decimally.

4. Find the face of a bill on Manchester, England, bought for $7500, exchange at 4.86.

5. Find the face of a bill on Frankfort, bought for $395.75, exchange at 95⅛.

6. Find the face of a bill on Geneva, Switzerland, bought for $4856, exchange at 5.22¼.

7. Find the face of a bill on Amsterdam, bought for $3750.67, exchange at 42½.

8. Find the face of a bill on Berlin, bought for $4000, exchange being 93¾.

9. Find the face of a bill on Liverpool, bought for $9720, exchange being quoted at 4.86.

10. What is the face of a bill on London, that may be purchased in New York for $277.42, exchange being quoted at 4.85?

11. What is the face of a bill on Hamburg, bought in New Orleans for $4500, exchange being at 95?

12. An agent in Boston having $7536.30 due his employer in England, is directed to remit by a bill on Liverpool. What is the face of the bill which he can purchase for this money, *exchange selling* at 4.91¼?

13. A trader in London wishes to invest £2500 in merchandise in Lisbon. If he remits to his correspondent at Lisbon a bill purchased for this sum, at the rate of 4.51 milreis to the pound sterling, what sum in the currency of Portugal will the agent receive?

14. A draft on Dublin for £360 cost $1751.94. What was the rate of exchange?

15. A merchant in Baltimore having received an importation from Madeira, invoiced at 1500 milreis (1 milreis = 1000 reis), allows his correspondent in Madeira to draw on him for the sum necessary to cover the cost, exchange on the United States being in Madeira 931¼ reis to the dollar. How much would the merchant have saved by remitting a draft on Madeira, purchased at $1.065 per milreis?

16. What is the rate of exchange on Berlin when $213.50 is paid for 890 marks?

Arbitration of Exchange.

707. Arbitration of Exchange is the process of computing exchange between two places by means of one or more intermediate exchanges.

Notes. — 1. When there is only one intermediate exchange, the process is called *simple arbitration*; when there are two or more intermediate exchanges, the process is called *compound arbitration*.

2. The arbitrated price is generally either greater or less than the price of direct exchanges; and the object of arbitration is to ascertain the best route for making drafts or remittances.

708. There are always three methods of receiving money from a place, or of transmitting money to a place, by means of indirect exchange through one intervening place. Thus,

If A is to receive money from C through B:

1st. A may draw on B, and B draw on C;
2d. A may draw on B, and C remit to B;
3d. B may draw on C, and remit to A.

If A is to transmit money to C through B:

1st. A may remit to B, and B remit to C;
2d. A may remit to B, and C draw on B;
3d. B may draw on A, and remit to C.

Examples.

709. To compute indirect exchange.

1. A man in Albany, N. Y., paid a demand in Paris of 5400 francs, by remitting to Amsterdam at the rate of \$ .41½ per guilder, and thence to Paris at the rate of 2.15 francs per guilder. How much United States money was required?

OPERATION.

$$\begin{aligned} \$\ ? &= 5400 \text{ francs.} \\ 2.15 \text{ francs} &= 1 \text{ guilder.} \\ 1 \text{ guilder} &= \$\ .41\tfrac{1}{2}. \end{aligned}$$

$$\frac{5400 \times \overset{.083}{\cancel{.415}}}{\underset{.43}{\cancel{2.15}}} = \$\,1042.32 \quad \textit{Ans.}$$

SOLUTION. — We are to determine how much United States money is equal to 5400 francs, and to do this we must first find how many guilders there are in 5400 francs. Now, since 2.15 francs = 1 guilder, 5400 *divided* by 2.15 will give the number of guilders; and that quotient *multiplied* by \$.415 (the value of one guilder), will give the number of dollars. Hence, 5400 and .415 are multipliers and 2.15 is a divisor. The units of currency being canceled and the work being abridged also by canceling common factors, we have \$1042.32 +, the required sum.

2. A resident of Naples, having a bequest of \$8720 made him in Boston, orders the remittance to be made to his agent in London, who remits the proceeds to Naples, reserving his commission of ½% on the draft sent. If exchange on London is quoted at \$4.875 in Boston, and the rate between London and Naples is 25.53 lire to the pound sterling, how much does the man realize from his bequest?

OPERATION.

$$\begin{aligned} ? \text{ lire} &= \$\,8720. \\ \$\,4.875 &= £\,1. \\ £\,1 &= 25.53 \text{ lire.} \\ 1.005 &= 1 \end{aligned}$$

$$\frac{\overset{1744}{\cancel{8720}} \times \overset{8.51}{\cancel{25.53}}}{4.875 \times \underset{\underset{.067}{\cancel{.201}}}{\cancel{1.005}}} = 45438.77 \text{ lire} \quad \textit{Ans.}$$

SOLUTION. — We make the statement as in the first example, according to the given rates of exchange. Then since the agent is to deduct ½% commission on the face of the draft before the purchase, the number of pounds must be divided by 1.005 (§ 536, III). We place 1.005 on the left as a divisor and obtain by cancellation, multiplication, and division, 45438.77 lire as the proceeds of the exchange.

3. A merchant in Chicago directs his agent in Albany to draw upon Baltimore at 1% discount, for $1200 due from the sales of produce; he then draws upon the Albany agent, at 2% premium, for the proceeds, after allowing the agent to reserve ½% for his commission. What sum does the merchant realize from his produce?

OPERATION.

$$\begin{aligned} ?\ C &= 1200\ B. \\ 100\ B &= 99\ A. \\ 100\ A &= 102\ C. \\ 1 &= .995 \end{aligned}$$

$$\frac{\overset{3}{\cancel{1200}} \times 99 \times 102 \times .199}{\cancel{100} \times \underset{\underset{5}{\cancel{25}}}{\cancel{100}}} = \$1205.70\ \textit{Ans.}$$

SOLUTION. — According to the given rates of exchange, 100 dollars in Baltimore equal 99 dollars in Albany; and 100 dollars in Albany equal 102 dollars in Chicago; and since the unit of currency is the same in each place, being $1, we represent its exchange value in each town by the initial letter, and make the statement as in the other examples. Then, since the agent is to reserve ½% commission from the avails of his draft, we place 1 − .005 = .995, § 536, on the right as a multiplier, and obtain by cancellation $1205.70.

RULE. — I. *Represent the required sum by (?), with the proper unit of currency affixed, and place it equal to the given sum on the right.*

II. *Arrange the given rates of exchange so that in any two consecutive equations the same unit of currency shall stand on opposite sides.*

III. *When there is commission for drawing, place* 1 *minus the rate on the left if the cost of exchange is required, and on the right if proceeds are required; and when there is commission for remitting, place* 1 *plus the rate on the right if cost is required, and on the left if proceeds are required.*

IV. *Divide the product of the numbers on the right by the product of the numbers on the left, canceling equal factors; the result will be the answer.*

Notes. — 1. Commission for drawing is commission on the *sale* of a draft; commission *for remitting* is commission on the *purchase* price of a draft.

2. The above method is sometimes called the *chain rule*, or *conjoined proportion*.

[redacted]

4. When exchange at New York on Paris is 5 francs 16 centimes per $ 1, and at Paris on Hamburg 1.23 francs per mark, what will be the arbitrated price in New York of 7680 marks on Hamburg?

5. A man in Philadelphia wishes to deposit $ 5000 in a bank at Stockholm, by remitting to Liverpool and thence to Stockholm. If the rate of exchange on Liverpool is 4.91 in Philadelphia, and the rate between Liverpool and Stockholm is 18½ crowns to £ 1, how much money will the man have in bank at Stockholm, allowing the agent at Liverpool ¼% for remitting?

6. A man in Cleveland wishes to draw on New Orleans for a bank stock dividend of $ 750, and exchange direct on New Orleans is 1¼% discount. How much will he save by drawing on his agent in New York at 1½% premium, allowing his agent to draw on New Orleans at 1% discount, brokerage at ½% ?

7. A man in Boston drew on Amsterdam for 6000 guilders at $ .415 per guilder. How much more would he have received if he had ordered remittance to London, and thence to New York, exchange at Amsterdam on London being 11.19 guilders per £ 1, and at London on New York 4.88, brokerage at 1¼% in London for remitting?

8. If at Philadelphia exchange on Liverpool is 4.89½, and at Liverpool on Paris 24 francs 96½ centimes per £ 1, what is the arbitrated rate of exchange between Philadelphia and Paris, through Liverpool?

9. An American resident of Amsterdam wishing to obtain funds from the United States to the amount of $ 6400, directs his agent in London to draw on the United States and remit the proceeds to him in a draft on Amsterdam, exchange on the United States being at 4.85 in London, and the rate between London and Amsterdam being 18*d.* per guilder. If the agent charges commission at ½% both for drawing and remitting, how much better is this arbitration than to draw directly on the United States at 41 cents per guilder?

AVERAGE OR EQUATION OF PAYMENTS.

710. **Equation of Payments** is the process of finding the time when two or more debts due at different times may be paid at once, without loss to either debtor or creditor.

711. The **Term of Credit** is the time that must elapse before a debt becomes due.

712. The **Average Term of Credit** is the time that must elapse before several debts, due at different times, may all be paid at once, without loss to debtor or creditor.

713. The **Equated Time** is the date at which the several debts may be canceled by one payment.

714. To **Average an Account** is to find the mean or equitable time of payment of the balance.

715. A **Focal Date** is a date with which all the others are compared in averaging an account.

NOTES. — 1. Any date may be taken as a focal date, but it is usual to take either the earliest or latest date at which any of the debts become due as the standard.

2. Each item of a book account draws interest from the time it is due, which may be either at the date of the transaction, or after a specified term of credit.

716. In averaging, there are two kinds of equations, **Simple** and **Compound**.

A **Simple Equation** is the process of finding the average time when the account contains only one side, which may be either a debit or credit side.

A **Compound Equation** is the process of averaging when both debts and credits are to be considered.

717. The computations are based on the following principles:

PRINCIPLES. — I. *Any debt is subject to legal interest from the time it is due.*

II. *The interest of a sum for a given time equals the interest of twice the sum for half the time, etc.*

III. *The payment of a sum before it is due is offset by keeping an equal sum an equal time after it is due.*

Examples.

718. To find the equated time when all the terms of credit begin at the same date.

1. In settling with a creditor on the first day of April, I find that I owe him $ 12 due in 5 months, $ 15 due in 2 months, and $ 18 due in 10 months. At what time may I pay the whole amount?

OPERATION.

$$\begin{aligned}
\$12 \times 5 &= 60\\
15 \times 2 &= 30\\
18 \times 10 &= 180\\
\hline
\$45 \qquad & \quad 270
\end{aligned}$$

$270 \div 45 = 6$ mo., average time of credit,
April 1, + 6 mo. = Oct. 1 *Ans.*

SOLUTION. — The whole amount to be paid, as is seen in the operation, is $ 45; and we are to find how long it shall be withheld, or what term of credit it shall have, as an equivalent for the various terms of credit on the different items. Now the value of credit on any sum is measured by the product of the money and time. Therefore, the credit on $ 12 for 5 months = the credit on $ 60 for 1 month, because $12 \times 5 = 60 \times 1$. In like manner, we have the credit on $ 15 for 2 months = the credit on $ 30 for 1 month; and the credit on $ 18 for 10 months = the credit on $ 180 for 1 month. Hence, by addition, the value of the several terms of credit on their respective sums equals a credit of 1 month on $ 270; and this equals a credit of 6 months on $ 45, because $45 \times 6 = 270 \times 1$; hence the equated time is 6 months after April 1 or Oct. 1.

RULE. — I. *Multiply each payment by its term of credit, and divide the sum of the products by the sum of the payments; the quotient will be the average term of credit.*

II. *Add the average term of credit to the date at which all the credits begin; the result will be the equated time of payment.*

NOTES. — 1. The periods of time used as multipliers must all be of the same denomination, and the quotient will be of the same denomination as the terms of credit; if these are months, and there is a remainder after the division, continue the division to days by reduction, always taking the nearest unit in the last result.

2. The several rules in equation of payments are based upon the principle of bank discount; for they imply that the *discount* of a sum paid before it is due equals the *interest* of the same sum paid after it is due.

2. On the first day of January, 1892, a man gave 3 notes, the first for $ 500 payable in 30 days; the second for $ 400 payable in 60 days; the third for $ 600 payable in 90 days.

What was the average term of credit, and what the equated time of payment?

3. I owe \$480 payable in 90 days, and \$320 payable in 60 days. My creditor consents to an extension of time to 1 year, and offers to take my note for the whole amount on interest at 6% from the equated time, or a note for the present worth of both debts, on interest from date. How much shall I gain if I choose the latter condition?

4. A man purchased real estate, and agreed to pay $\frac{1}{3}$ of the price in 3 months, $\frac{1}{4}$ in 8 months, and the remainder in 1 year. Wishing to cancel the whole obligation at a single payment, how long may this payment be deferred?

5. I bought merchandise April 1, as follows: \$280 on 3 months, \$300 on 4 months, \$200 on 5 months, \$560 on 6 months. What is the equated time of payment?

719. To find the equated time when the terms of credit begin at different dates and the account has only one side.

1. When does the amount of the following bill become due, by averaging?

CHARLES CROSBY,

1896.		To BRONSON & Co., *Dr.*
Jan. 12.	To Mdse.	\$400
" 16.	" Mdse. on 2 mo.	600
Apr. 20.	" Cash.	375

FIRST OPERATION.

Due.	Da.	Items.	Prod.
Jan. 12		400	
Mar. 16	64	600	38400
Apr. 20	99	375	37125
		1375	75525

75525 ÷ 1375 = 55 days.

Ans. { 55 days after Jan. 12, or Mar. 7.

SECOND OPERATION.

Due.	Da.	Items.	Prod.
Jan. 12	99	400	39600
Mar. 16	35	600	21000
Apr. 20	0	375	
		1375	60600

60600 ÷ 1375 = 44 days.

Ans. { 44 days before Apr. 20, or Mar. 7.

[redacted]

SOLUTION. — The three items of the bill are due Jan. 12, March 16, and April 20, respectively. In the first operation we use the *earliest* maturity, Jan. 12, for a focal date, and find the difference in days between this date and each of the others, taking into account the exact number of days in each calendar month; thus, from Jan. 12 to March 16 is 64 days (in leap year); from Jan. 12 to April 20 is 99 days. Hence, from Jan. 12 the first item has no credit, the second has 64 days' credit, and the third 99 days' credit, as appears in the column marked *da.* We now proceed to find the products as in § 718, whence we obtain the average credit, 54.9 or 55 days, and the equated time, March 7.

In the second operation, the *latest* maturity, April 20, is taken for a focal date, and the work may be explained thus: Suppose the account to be settled April 20. At that time the first item has been due 99 days, and must therefore draw interest for this time. But interest on $ 400 for 99 days = the interest on $ 39600 for 1 day. The second item must draw interest 35 days; but interest on $ 600 for 35 days = interest on $ 21000 for 1 day. Taking the sum of the products, we find that the whole amount of interest due April 20 = the interest on $ 60600 for 1 day; and this is found, by division, equal to the interest on $ 1375 for 44 days, which is the average term of interest. Hence the account would be settled April 20, by paying $ 1375, with interest on the same for 44 days. This shows that $ 1375 has been used 44 days, that is, it falls due March 7, *without interest.*

RULE. — I. *Find the time at which each item becomes due, by adding to the date of each transaction the term of credit, if any is specified, and write these dates in a column.*

II. *Assume either the earliest or the latest date for a focal date; find the difference in days between the focal date and each of the other dates, and write the results in a second column.*

III. *Write the items of the account in a third column, and multiply each by the corresponding number of days in the preceding column, writing the products in a fourth column.*

IV. *Divide the sum of the products by the sum of the items. The quotient will be the average term of credit, when the earliest date is the focal date, or the average term of interest, when the latest date is the focal date. In either case, reckon from the focal date* TOWARD *the other dates, to find the equated time of payment.*

NOTES. — 1. When dollars and cents are given, it is generally sufficient to take only dollars in the multiplicand, rejecting the cents when less than 50, and carrying 1 to the dollars, if the cents are more than 50.

2. Months in any terms of credit are understood to be calendar months; the time must therefore be carried forward to the same day of the month in which the term of credit expires.

2. Find the equated time of payment of the following bill:

James Gordon,

1896.						To Henry Lancey, *Dr.*		
Mar.	4.	To	100 yd.	Cassimere,	@ $2.50	. . . . .		$250
"	25.	"	3000 "	French Prints,	" .12	. . . . .		360
Apr.	16.	"	1200 "	Sheeting,	" .08	. . . . .		96
"	30.	"	400 "	Oil Cloth,	" .50	. . . . .		200
May	17.	"	Sundries	. . . . .	. . . . .	. . . . .		350
June	1.	"	100 yd.	Cassimere,	@ 2.50	. . . . .		250

3. I sell goods to A at different times, and for different terms of credit, as follows:

Sept. 12, 1895,	a bill on	1 mo.	credit, for	. . . .	$180
Oct. 7,	" "	1 "	"	. . . .	300
Nov. 16,	" "	2 "	"	. . . .	150
Dec. 20,	" "	3 "	"	. . . .	350
Jan. 25, 1896,	"	1 "	"	. . . .	130
Jan. 28,	" "	1 "	"	. . . .	200
Feb. 24,	" "	1 "	"	. . . .	140

If I take his note in settlement, at what time shall interest commence?

4. What is the average of the following account?

1896,	Oct. 1.	Mdse., on 60 days	. . . .	$240
"	Nov. 12.	" " "	. . . .	500
"	Dec. 18.	" " "	. . . .	436
1897,	Jan. 16.	" " "	. . . .	325
"	Feb. 24.	" " "	. . . .	436
"	Mar. 17.	" " "	. . . .	537
"	Mar. 20.	" " "	. . . .	500
"	Apr. 15.	" " "	. . . .	600

5. I have 4 notes, as follows: The first for $350, due Aug. 16, 1895; the second for $250, due Oct. 15, 1895; the third for $300, due Dec. 14, 1895; the fourth for $248, due Feb. 12, 1896. When shall a note for which I may exchange the four be made payable?

720. To find the equated time when the terms of credit begin at different times, and the account has both a debit and a credit side.

1. Find at what date the following account may be paid, without loss to either debtor or creditor:

Dr. JOHN LYMAN. *Cr.*

1896.				1896.			
June 12	To Mdse.	530	00	June 24	By Draft at 30 da.	480	00
Sept. 12	" "	428	00	Aug. 20	" Cash,	230	00
Oct. 28	" Sundries,	440	00	Oct. 8	" "	140	00

Dr. OPERATION. *Cr.*

	Due.	Da.	Items.	Products.	Due.	Da.	Items.	Products.
	June 12	138	530	73140	July 27	93	480	44640
	Sept. 12	46	428	19688	Aug. 20	69	230	15870
Focal date,	Oct. 28	0	440		Oct. 8	20	140	2800
			1398	92828			850	63310
			850	63310				
	Balances,		548	29518				

29518 ÷ 548 = 54 da., average term of interest.
Oct. 28 – 54 da. = Sept. 4, balance due *Ans.*

SOLUTION. — In this operation we have written the dates of maturity on either side, allowing three days' grace to the draft. The latest date, Oct. 28, is assumed as the focal date for *both sides*, and the two columns marked *da.* show the difference in days between the focal date and each of the other dates. The products are obtained as in § 719, and the balance is found between the items on the two sides, and also between the products.

These balances, being both on the Dr. side, show that John Lyman on the day of the focal date, Oct. 28, owes $548, with interest on $29518 for 1 day. By division, this interest is found to be equal to the interest on $548 for 54 days. Hence this balance, $548, has been due 54 days; and reckoning back from the focal date, we obtain the equated time of payment, Sept. 4.

Had we taken the *earliest* maturity, June 12, for the focal date, we should have obtained 84 days for the interval of time; and since in this case the products would represent the *credit* to which the several items are entitled *after* June 12, we should *add* 84 days to the focal date, which would give Sept. 4, as before.

2. What is the balance of the following account and when is it due?

Dr. CHARLES DERBY. *Cr.*

1895.				1895.			
Jan. 21	To Mdse.	32	00	Jan. 1	By Cash,	84	00
Mar. 5	" "	145	00	Feb. 4	" "	40	00
" 22	" "	194	00	Mar. 30	" "	12	00

Dr. OPERATION. *Cr.*

Due.	Da.	Items.	Products.	Due.	Da.	Items.	Products.
Jan. 21	68	32	2176	Jan. 1	88	84	7392
Mar. 5	25	145	3625	Feb. 4	54	40	2160
" 22	8	194	1552	Mar. 30	0	12	
		371	7353			136	9552
		136					7353
Balance of account,		235		Balance of products,			2199

$2199 \div 235 = 9$ da.; Mar. 30 + 9 da. = Apr. 8 *Ans.*

SOLUTION. — We take the latest maturity, March 30, for the focal date, and consequently the products represent the *interest* due upon the several items, at that date. We find the balance of the items upon the Dr. side, and the balance of the products upon the Cr. side. The debtor therefore owes, on March 30, $235, but is entitled to such a term of interest on the same as will be equivalent to the interest on $2199 for 1 day, which by division, is found to be 9 days. Hence the balance is due March 30 + 9 da. = April 8. Thus we see that when the balances are on opposite sides, the interval of time is counted *from* the other dates. If we take, in this example, the *earliest* date for the focal date, the balances will both be upon the Dr. side, and the interval of time will be 97 days, which, reckoned forward from the focal date, will give the equated time as before.

RULE. — I. *Find the time when each item of the account is due, and write the dates, in two columns, on the sides of the account to which they respectively belong.*

II. *Use either the earliest or the latest of these dates as the focal date for both sides, and find the products as in* § 719.

III. *Divide the balance of the products by the balance of the account; the quotient will be the interval of time, which must be reckoned from the focal date* TOWARD *the other dates when both balances are on the same side of the account, but* FROM *the other dates when the balances are on opposite sides of the account.*

NOTES. — 1. Instead of the products, we may obtain the interest, at any per cent, on the several items for the corresponding intervals of time, and divide the balance of interest by the interest on the balance of the account for 1 day; the quotient will be the interval of time to be added to, or subtracted from, the focal date, according to the rule. The time obtained will be the same, at whatever rate the interest may be computed.

2. There may be such a combination of debits and credits, that the equated time will be earlier or later than any date of the account.

3. Find the average maturity of the following account:

Dr. A. B. ARMOUR. *Cr.*

1897.				1897.			
Feb. 12	To Mdse.	85	75	Mar. 15	By Bal. old acc't	97	86
" 25	" "	86	24	April 17	" Cash	56	00
April 16	" "	174	96	May 25	" "	25	00
May 20	" "	94	78	June 8	" Sundries	94	75

Dr. OPERATION. *Cr.*

Due.	Da.	Items.	Interest.	Due.	Da.	Items.	Interest.
Feb. 12	116	85.75	1.66	Mar. 15	85	97.86	1.38
" 25	103	86.24	.62	April 17	52	56.00	.49
April 16	53	174.96	1.55	May 25	14	25.00	.06
May 20	19	94.78	.30	June 8		94.75	
		391.73	4.13			273.11	1.93
		273.11	1.93				
Balances,		118.62	2.20				

Int. on $118.62 for 1 da. = $.0198.

$2.20 \div .0198 = 111$ da.; June 8 − 111 da. = Feb. 17, 1897 *Ans.*

SOLUTION. — Taking the latest maturity, June 8, for the focal date, we find the *interest* of each item, at 6%, from its maturity to the focal date; then, taking the balance, we find the interest due on the account to be $2.20. Dividing this interest by the interest on the balance of the items for 1 day, we obtain 111 days, the time required for the interest, $2.20, to accrue. The average maturity, therefore, is June 8 − 111 da. = Feb 17, 1897. It is evident that when the balances occur on opposite sides, the interval of time will be reckoned as in the method by products.

4. Find the average maturity of the following account:

Dr. THOMAS LARDNER. *Cr.*

1896.				1896.			
March 1	To Sundries,	486	00	March 25	By Draft, at 60 da.	400	00
April 12	" Mdse.	548	00	April 6	" " 80 "	650	00
July 16	" "	812	00	June 20	" Cash,	200	00
Sept. 14	" "	586	00	Aug. 8	" "	84	00

5. Find the average maturity of the following account:

Dr. DAVID SANFORD. *Cr.*

1895.				1895.			
Feb. 1	To Mdse. on 3 mo.	54	36	April 1	By Cash,	50	00
" 12	" " " 2 "	28	45	May 16	" Draft, at 30 da.	30	00
March 16	" Sundries,	95	75	June 12	" "	125	00
June 25	" Mdse.	26	32	" 20	" Cash	150	00

6. If the following account were settled April 6, 1894, by draft on time, how many days' credit should be given?

Dr. OLIVER WAINWRIGHT. *Cr.*

1894.				1894.			
Feb. 1	To Mdse.	36	72	Jan. 10	By Cash,	98	72
March 1	" "	48	25	" 21	" "	25	84
March 17	" "	72	36	March 23	" Sundries,	15	17
April 1	" "	93	48	April 6	" "	8	96

7. Find the average maturity of the following account:

Dr. JOHN WOLCOTT. *Cr.*

1895.				1895.			
Feb. 1	To Mdse.	448	00	Feb. 20	By Amt. brought for'd	560	00
" 4	" Cash	364	00	" 23	" 1 Carriage.	264	00
" 20	" "	232	00	" 25	" Cash	900	00

8. I owe $1000 due April 25. If I pay $560 April 1, and $324 April 21, when, in equity, should I pay the balance?

NOTE. — Make the $1000 the Dr. side of an account, and the payments the Cr. side, and then average.

9. A man owes $684, payable Aug. 12, and $468, payable Oct. 15. If he pays $839, Aug. 1, what will be the equated time for the payment of the balance?

10. A man holds 3 notes, the first for $500, due March 1, the second for $800, due June 1, and the third for $600, due Aug. 1. He wishes to exchange them for two others, one of which shall be for $1000, payable April 1. What will be the face and when the maturity of the others?

11. A owes $500, due April 12, and $1000, due Sept. 20, and wishes to discharge the obligation by two equal payments, made at an interval of 60 days. When must the two payments be made?

[redacted]

ACCOUNT SALES.

721. An **Account Sales** is an account rendered by a commission merchant of goods sold on account of a consignor, and contains a statement of the sales, the attendant charges, and the net proceeds due the owner.

NOTE. — Expenses include freight, cartage, storage, insurance, etc.

722. **Guaranty** is a charge made in addition to commission, for securing the owner against the risk of non-payment, in case of goods sold on credit.

723. **Storage** is a charge made for keeping the goods, and may be reckoned by the week or month, on each article or piece.

724. **Primage** is an allowance which is paid by a shipper or consignor of goods to the master and sailors of a vessel, for loading it.

725. A commission merchant having sold a shipment of goods by parts at different times, and on various terms, makes a final settlement by deducting all the charges, and by accrediting the owner with the net proceeds. It is evident therefore, that:

I. The commission and guaranty should be accredited to the agent at the average maturity of the *sales*.

II. The net proceeds should be accredited to the consignor at the average maturity of the entire account.

RULE. — I. To compute the storage. — *Multiply each article or parcel by the time it is in store, and multiply the sum of the products by the rate per unit; the result will be the storage.*

II. To find when the net proceeds are due. — *Average the sales alone, and the result will be the date to be given to the commission and guaranty; then make the sales the Cr. side, and the charges the Dr. side, and average the entire account by the method shown in* § 720.

NOTE. — In averaging, either the product method or the interest method may be used.

Examples.

726. 1. Find the net proceeds of the following account, and when due.

Account sales of 1000 barrels of flour received May 1, 1895, by John Fisk from Tyler, Bell & Co.

1895.					
June	3	Sold 200 bbl. at $3.00		$ 600.00	
June	30	" 350 " 3.25 on 1 mo.		1137.50	
July	29	" 400 " 3.15		1260.00	
Aug.	6	" 50 " 3.75		187.50	
		CHARGES.		$ 3185.00	
May	1	To freight, cartage, primage, and cooperage	$ 250.25		
		" insurance	55.50		
			$ 305.75		
Aug.	6	To storage from May 1 at 2 cts. a wk.			
		On 200 bbl. 5 wk. 1000 wk.			
		" 350 " 9 " 3150 "			
		" 400 " 13 " 5200 "			
		" 50 " 14 " 700 "			
		10050 wk. at 2 cts. =	$ 201.00		
		To Commission on $3185 at 2½ %	$ 79.63		
		" Guaranty on $1137.50 at 2½%	28.44		
			$ 108.07	$ 614.82	
				$ 2570.18	

Dr. OPERATION. *Cr.*

CHARGES.				SALES.			
Due.	Da. from May 1.	Items.	Products.	Due.	Da. from May 1.	Items.	Products.
May 1	00	305.75	00	June 3	33	600.00	19800.00
Aug. 6	97	201.00	19497.00	July 30	90	1137.50	102375.00
July 20	80	108.07	8645.60	July 29	89	1260.00	112140.00
				Aug. 6	97	187.50	18187.50
		614.82	28142.60			3185.00	252502.50
						614.82	28142.60
						2570.18	224359.90

252502.50 ÷ 3185 = 80 da.

May 1 + 80 da. = July 20, date for commission and guaranty.

224359.90 ÷ 2570.18 = 88 da.

May 1 + 88 da. = July 28, 1895, proceeds due }
Proceeds = $2570.18. } *Ans.*

[redacted]

SOLUTION. — We first find the proceeds by deducting the charges from the amount of sales, computing the storage according to the rule. Then we make the sales the Cr. side, and equate the time by § 719, taking May 1, the earliest date, as the focal date. The answer, July 20, is the date for commission and guaranty. As the dates for the other charges are known, we proceed to equate the time for charges on the Dr. side, and to balance the two sides as in § 720, and we find the proceeds to be $ 2570.18, due July 28, 1895.

NOTE. — The time for which storage is charged on each part of the shipment is the interval, reduced to weeks, between May 1, when the flour was received into store, and the date of sale. Every fraction of a week is reckoned a full week.

2. Frank Aldrich sold on account of F. Grant & Co. a consignment of 1000 bu. of wheat as follows: Feb. 1, 1895, 200 bu. @ $ .62, cash; March 1, 1895, 800 bu. @ $ .65 on 2 mo. The charges were: Jan. 1, freight, cartage, etc., $ 75; storage from Jan. 5, @ $ .01 a bu.; insurance $ 5.00; commission and guaranty $2\frac{1}{2}\%$. Find the proceeds and when due.

3. Henry Osgood sold on account of E. L. Curry & Bro. of Brooklyn, the following: March 1, 1894, 6000 yd. black ribbon @ $ .50; March 15, 2000 yd. navy ribbon @ $ .15, and 1000 yd. brown at $ .25. The charges were: Feb. 1, freight and cartage $ 3.75; Feb. 6, advertising $ 5; commission 2%. Find the proceeds and when due.

4. A commission merchant in Boston received into his store on May 1, 1895, 1000 bbl. of flour, paying as charges on the same day, freight $ 175.48, cartage $ 56.25, and cooperage $ 8.37. He sold out the shipment as follows: June 3, 200 bbl. @ $ 4.25; June 30, 350 bbl. @ $ 4.50; July 29, 400 bbl. @ $ $4.12\frac{1}{2}$; Aug. 6, 50 bbl. @ $ 4.00. Required the net proceeds, and the date when they shall be accredited to the owner, allowing commission at $3\frac{1}{2}\%$, and storage at $ .02 per week per bbl.

5. John Anderson sold on account of J. B. Walbridge & Co. of Philadelphia, the following: April 1, 1896, 2000 yd. silk @ $ 1.25; May 15, 6000 yd. cassimere @ $ 1.00, and 2000 yd. silk @ $ 1.25. The charges were: March 1, freight and cartage $ 25; March 10, advertising $ 20; commission 2%. *Find the proceeds* and when due.

SETTLEMENT OF ACCOUNTS CURRENT.

Examples.

727. To find the cash balance of an account current at any given date.

1. Required the cash value of the following account, July 1, 1896, interest at 6%.

Dr. J. BURNS *in account current with* TYLER & CO. *Cr.*

1896.				1896.			
Feb. 25	To Mdse. on 3 mo.	360	75	Mar. 1	By Cash on acct.	250	00
Mar. 20	" " " 3 "	240	56	April 20	" Accept. at 30 da.	300	00
April 26	" " " 3 "	875	24	June 12	" Sundries.	375	00
June 24	" " " 2 "	285	25	" 27	" Cash on acct.	400	00

Dr. OPERATION. *Cr.*

Due.	Da.	Items. Int.	Cash val.	Due.	Da.	Items. Int.	Cash val.
May 25	37	360.75 + 2.22	362.97	Mar. 1	122	250.00 + 5.08	255.08
June 20	11	240.56 + .44	241.00	May 20	42	300.00 + 2.10	302.10
July 26	25	875.24 − 3.65	871.59	June 12	19	375.00 + 1.19	376.19
Aug. 24	54	285.25 − 2.12	283.13	" 27	4	400.00 + .27	400.27
			1708.69				1333.64

$ 1708.69 − $1333.64 = $ 375.05 *Ans.*

SOLUTION. — For either side of the account we write the dates at which the several items are due, and the days intervening between these dates and the day of settlement, July 1. We then compute the interest on each item for the corresponding interval of time, and add it to the item if the maturity is *before* July 1, and subtract it from the item if the maturity is after July 1; the results must be the cash values of the several items on July 1. Adding the two columns of cash values, and subtracting the less sum from the greater, we have $ 375.05 the *cash balance* required.

RULE. — I. *Find the number of days intervening between each maturity and the day of settlement.*

II. *Compute the interest on each item for the corresponding interval of time; add the interest to the item if the maturity is before the day of settlement, and subtract it from the item if the maturity is after the day of settlement; the result will be the cash values of the several items.*

III. *Add each column of cash values, and the difference of the amounts will be the cash balance required.*

2. Find the cash balance of the following account on June 1, 1894, interest at 6 per cent.

Dr. ALVAN PARKE, *in account current with* C. D. CALL & Co. *Cr.*

1894.				1894.			
Jan. 12	To Check,	500	86	Feb. 1	By Bal. from old acct.	586	72
" 26	" "	250	48	" 8	" Cash,	486	57
Feb. 13	" "	400	00	March 26	" "	1260	78
March 16	" "	750	00	April 20	" "	756	86
April 25	" "	200	00	May 12	" "	248	79

3. What is the cash balance of the following account on Dec. 31, 1895, at 7 per cent?

Dr. JAMES HANSON. *Cr.*

1895.				1895.			
Sept. 8	To Sundries,	478	36	Sept. 17	By Sundries,	96	54
Oct. 2	" Mdse. on 3 mo.	256	37	" 20	" Cash on acct.	200	00
" 21	" " " 3 "	375	26	Oct. 8	" " "	325	00
Nov. 12	" " " 3 "	80	00	Nov. 17	" " "	50	00
Dec. 15	" Sundries,	148	76	Dec. 27	" " "	84	00

4. Find the cash balance of the following account on Dec. 3, 1896, interest at 6%.

Dr. JAMES OSGOOD, *in account current with* J. F. MILLER & Co. *Cr.*

1896.				1896.			
Feb. 3	To Mdse.	500	75	March 1	By Cash,	800	00
Feb. 16	" " on 3 mo.	620	00	April 3	" "	50	00
March 8	" " " 3 "	850	00	May 15	" "	115	75
April 15	" " " 3 "	240	75	July 6	" "	200	00
May 3	" " " 1 "	819	50	July 15	" "	800	00
Aug. 15	" " " 3 "	625	25	Sept. 1	" "	500	00
Sept. 8	" " " 2 "	814	05				

5. Find the cash balance of the following account on Jan. 15, 1896, interest at 5%.

Dr. FRANK ALLEN *in account current with* HARVEY & BALES. *Cr.*

1896.				1896.			
April 5	To Sundries,	250	00	May 10	By Cash,	150	00
May 5	" Mdse.	120	00	June 10	" "	200	00
June 5	" "	15	75	Aug. 10	" "	10	00
July 5	" "	650	50	Sept. 15	" "	15	00
Aug. 5	" "	400	50	Oct. 15	" "	800	00
Sept. 5	" "	750	00	Dec. 15	" "	500	00

INVOLUTION.

728. A **Power** is the product arising from multiplying a number by itself, or repeating it any number of times as a factor (§ 104).

729. **Involution** is the process of raising a number to a given power (§ 110).

730. A **Perfect Power** is a number that has an exact root, and an **Imperfect Power** is one that has not an exact root.

731. To discover the principles which govern the process of involution.

Suppose we wish to find the sixth power of 4.

$$4 \times 4 \times 4 \times 4 \times 4 \times 4 = 4^6 = 4096.$$

Or, $(4 \times 4) \times (4 \times 4) \times (4 \times 4) = 4^2 \times 4^2 \times 4^2 = 4096.$

Or, $(4 \times 4 \times 4) \times (4 \times 4 \times 4) = 4^3 \times 4^3 = 4096.$

Or, $(4 \times 4 \times 4 \times 4) \times (4 \times 4) = 4^4 \times 4^2 = 4096.$

Or, $4 \times (4 \times 4 \times 4 \times 4 \times 4) = 4 \times 4^5 = 4096.$

In these examples we see that the sum of the exponents in the various multiplications are equal to the required exponent and the results are all alike. Thus, $2 + 2 + 2 = 6$, $3 + 3 = 6$, $4 + 2 = 6$, $5 + 1 = 6$, etc. We also see that the sixth power is equal to the cube of the square (that is, the square taken three times in continued multiplication) or the square of the cube.

732. From these facts we derive the following principles.

PRINCIPLES.—I. *The exponent of any power is equal to the number of times the root has been taken as a factor in continued multiplication.*

II. The PRODUCT *of any two or more powers of the same number is the power denoted by the* SUM *of their exponent.*

[redacted]

III. *If any power of a number is raised to any given* POWER, *the result will be that power of the number denoted by the* PRODUCT *of the exponents.*

Examples.

733. To find any power of a number.

1. What is the fifth power of 6?

OPERATION.

$6 \times 6 \times 6 \times 6 \times 6 = 7776$ *Ans.*

Or,

$6 \times 6 = 6^2 = 36$

$36 \times 6 = 6^3 = 216$

$6^3 \times 6^2 = 6^5 = 216 \times 36 = 7776$ *Ans.*

SOLUTION. — We multiply 6 by itself, and this product by 6, and so on, until 6 has been taken 5 times in continued multiplication; the final product, 7776, is the power required (§ 732, I). Or, we may first form the second and third powers; then the product of these two powers will be the fifth power required (§ 732, II).

2. What is the sixth power of 12?

OPERATION.

$12^2 = 144$

$144^3 = 2985984$ *Ans.*

SOLUTION. — We find the cube of the second power, which must be the sixth power (§ 732, III).

RULE. — I. *Multiply the given number by itself in continued multiplication, till it has been taken as many times as a factor as there are units in the exponent of the required power.* Or,

Multiply together two or more powers of the given number, the sum of whose exponents is equal to the exponent of the required power. Or,

Raise some power of the given number to such a power that the product of the two exponents shall be equal to the exponent of the required power.

NOTES. — 1. The number of multiplications will be *one less* than the exponent, since the root in the first multiplication is used *twice*, once as multiplicand and once as multiplier.

2. A fraction is involved to any power by involving each of its terms separately to the required power.

3. Mixed numbers should be reduced to improper fractions or decimals before involution is performed.

4. When the number to be involved is a decimal, contracted multiplication may be applied with great advantage.

Raise the following numbers to the powers indicated by their exponents:

3. 79^2.	6. 1450^2.	9. $.437^3$.
4. 85^3.	7. $16\frac{4}{5}^4$.	10. 1.05^6.
5. 25.4^3.	8. 2^{20}.	11. $.009^5$.

Find the value of each of the following expressions:

12. 4.367^4.	15. $4.6^3 \times 25^3$.	18. $7^8 \div 3.08$.
13. $(\frac{7}{8})^3$.	16. $(6\frac{3}{4})^4 - 7.25^2$.	19. $4^3 \times 5^6 \times 12^3$.
14. $(2\frac{3}{4})^5$.	17. $\frac{7}{8}$ of $(\frac{4}{5})^3$ of $(3\frac{4}{7})^2$.	20. $4^2 \times 10^4 \times 3^2$.

21. $(4^3 \times 5^6 \times 12^3) \div (4^3 \times 10^2 \times 6^2)$.

22. $(3^5 \times 2^3 \times 4^3) \div (2^2 \times 3^3 \times 4^2)$.

23. $(2^5 \times 3^4) + (3^2 \times 2^4) - (2^3 \times \frac{1}{2}^2)$.

734. To find the square of a number in terms of its tens and units.

1. Find the square of 23 in terms of its tens and units.

OPERATION.

$$\begin{array}{rl}
23 = & 20 + 3 \\
23 = & 20 + 3 \\
\hline
69 = & 20 \times 3 + 3^2 \\
46 \;= & 20^2 + 20 \times 3 \\
\hline
529 = & 20^2 + 2\,(20 \times 3) + 3^2
\end{array}$$

SOLUTION. — $23 = 20 + 3$. Multiplying $20 + 3$ by 3 and indicating the operation, we have 20×3 and 3×3, or 3^2. Multiplying $20 + 3$ by 20, we have 20×20, or 20^2, and 20×3. Adding the partial products, the result is $20^2 + 2$ times $20 \times 3 + 3^2$, which is equal to 529.

RULE. — To find the square of a number consisting of tens and units. — *To the square of the tens add twice the product of the tens by the units and the square of the units.*

NOTE. — When a number is separated into *any* two parts, its square is always equal to the square of the first part + twice the product of the first by the second + the square of the second part. Thus, $23 = 12 + 11$; and $23^2 = 12^2 + 2\,(12 \times 11) + 11^2 = 529$.

Hence, the rule for squaring any number by tens and units may be expressed by the formula: $t^2 + 2tu + u^2$.

In the same way find the square of:

2. 36.	5. 58.	8. 69.	11. 109.
3. 65.	6. 92.	9. 97.	12. 157.
4. 39.	7. 78.	10. 73.	13. 275.

[redacted]

735. To find the cube of a number in terms of its tens and units.

1. Find the cube of 23 in terms of its tens and units.

OPERATION.

$$
\begin{array}{rcl}
23 & = & 20 + 3 \\
23 & = & 20 + 3 \\
\hline
69 & = & 20 \times 3 + 3^2 \\
46 & = & 20^2 + 20 \times 3 \\
\hline
529 & = & 20^2 + 2\,(20 \times 3) + 3^2 \\
23 & = & 20 + 3 \\
\hline
1587 & = & 20^2 \times 3 + 2\,(20 \times 3^2) + 3^3 \\
1058 & = & 20^3 + 2\,(20^2 \times 3) + (20 \times 3^2) \\
\hline
12167 & = & 20^3 + 3\,(20^2 \times 3) + 3\,(20 \times 3^2) + 3^3
\end{array}
$$

SOLUTION. — The cube of 23 = 23 × 23 × 23, or $23^2 \times 23$. We proceed to find the square as before, which is $20^2 + 2(20 \times 3) + 3^2$. Multiplying this first by 3 and then by 20 and adding these partial products, the result is $20^3 + 3(20^2 \times 3) + 3(20 \times 3^2) + 3^3$.

RULE. — To find the cube of a number consisting of tens and units. — *To the cube of the tens add three times the product of the square of the tens by the units, three times the product of the tens by the square of the units, and the cube of the units.*

NOTE. — The cube of a number divided into any two parts is equal to the cube of the first part + 3 times the square of the first part by the second part + 3 times the first part by the square of the second part + the cube of the second part. Thus, 23 = 12 + 11, and $23^3 = 12^3 + 3\,(12^2 \times 11) + 3\,(12 \times 11^2) + 11^3 = 12167$.

Hence, the rule may be expressed by a formula as follows: $t^3 + 3t^2u + 3tu^2 + u^3$.

In the same way find the cube of:

2. 27. 4. 52. 6. 115. 8. 274. 10. 613.
3. 42. 5. 79. 7. 140. 9. 569. 11. 996.

APPLICATIONS OF INVOLUTION.

736. The following principles of physics afford application for the rules of involution.

PRINCIPLES. — I. *The intensity of light varies inversely as the square of the distance from the source of illumination.*

II. *The intensity of sound varies inversely as the square of the distance from the source of the sound.*

III. *The heating effect of a small radiant mass upon a distant object varies inversely as the square of the distance.*

IV. *The force of attraction or repulsion exerted between two magnetic poles is inversely proportional to the square of the distance between them.*

V. *Gravitation varies inversely as the square of the distance between the centers of gravity.*

Examples.

737. 1. From a vessel on the ocean, a light in a lighthouse could be seen dimly 25 miles distant. How much brighter would the light appear, when the vessel was 5 miles distant?

2. I have 2 lamps, one of 4 candle power and one of 16 candle power; if the former is 20 ft. distant, how far away must I place the latter to give me the same amount of light?

3. A bell heard by A is heard by B 8 times as far distant. How loud will it sound to B as compared to A?

4. If the earth were removed to $\frac{1}{5}$ its distance from the sun, how much more intense would be the heat received by it?

5. A body 4000 miles from the center of the earth (or at the earth's surface) weighs 900 pounds. What would it weigh 12000 miles from the center of the earth?

6. A pistol shot is heard by A. B hears it $\frac{1}{25}$ as loud. How much further is B from the pistol than A?

7. Two magnetic poles $\frac{1}{4}$ of an inch apart have an attraction for each other whose force would lift a pound weight. What weight would they lift if they were an inch apart?

8. A wax taper is held a certain distance from a flame. It requires an amount of heat 16 times as great as that now acting upon it to ignite it. How much nearer must I move the taper if I wish to light it?

9. If an electric light and a gas light are 16 ft. apart, and the former gives 9 times as much light as the latter, at what point between the two must an object be placed so as to receive the same amount of light from each?

[redacted]

EVOLUTION.

738. A **Root** is a factor repeated to produce a power (§ 105); thus, in the expression $7 \times 7 \times 7 = 343$, 7 is the root from which the power, 343, is produced.

739. **Evolution** is the process of extracting the root of a number considered as a power; it is the reverse of Involution.

Any number whatever may be considered as a power whose root is to be extracted.

740. A **Rational Root** is a root that can be exactly obtained.

741. A **Surd** is an indicated root that cannot be exactly obtained because the power is imperfect.

742. The **Radical Sign** is the character, $\sqrt{\ }$, which, placed before a number, indicates that its root is to be extracted.

743. The **Index** of the root is the figure placed above the radical sign, to denote what root is to be taken. When no index is written, the index 2 is always understood.

744. The names of roots are derived from the corresponding powers, and are denoted by the indices of the radical sign. Thus, $\sqrt{100}$ denotes the *square root* of 100, $\sqrt[3]{100}$ denotes the *cube root* of 100; $\sqrt[4]{100}$ denotes the *fourth root* of 100; etc.

745. Evolution is sometimes denoted by a fractional exponent, the name of the root to be extracted being indicated by the denominator. Thus, the square root of 10 may be written $10^{\frac{1}{2}}$; the cube root of 10, $10^{\frac{1}{3}}$, etc.

746. **Fractional Exponents** are also used to denote both involution and evolution in the same expression, the numerator *indicating the* power to which the given number is to be

raised, and the denominator the root of the power which is to be taken; thus, $7^{\frac{2}{3}}$ denotes the cube root of the second power of 7, and is the same as $\sqrt[3]{7^2}$; so also $7^{\frac{5}{2}} = \sqrt{7^5}$.

747. In extracting any root of a number, any term or terms may be regarded as tens of the next inferior order. Thus, in 2546, the 2 may be considered as tens of the third order, the 25 as tens of the second order, or the 254 as tens of the first order.

EVOLUTION BY FACTORING.

Examples.

748. To find any root of a number by factoring.

1. Find the cube root of 1728.

OPERATION.

3	1728
3	576
3	192
2	64
2	32
2	16
2	8
2	4
	2

SOLUTION.—A number that is a perfect cube is composed of *three* equal factors, and one of them is the cube root of that number.

The prime factors of 1728 are 3, 3, 3, 2, 2, 2, 2, 2, 2; hence $1728 = (3 \times 2 \times 2) \times (3 \times 2 \times 2) \times (3 \times 2 \times 2)$; therefore the cube root of 1728 is $(3 \times 2 \times 2)$, or 12.

RULE.—*Resolve the given number into its prime factors; then, to produce the square root, take one of every two equal factors; to produce the cube root take one of every three equal factors; and so on.*

2. Find the square root of 64, 256, 576, 6561.
3. Find the cube root of 729, 2744, 9261, 3375.
4. Find the square root of 225, 256, 289.
5. Find the cube root of 4913, 8000, 24389.
6. Find the fourth root of 81, 256, 625, 1296.
7. Find the fifth root of 243, 1024, 3125.
8. Find the sixth root of 729, 4096, 15625.
9. Find the seventh root of 2187, 16384, 78125.

SQUARE ROOT.

749. The **Square Root** of a number is one of the two equal factors that produce the number. Thus, the square root of 64 is 8, for $8 \times 8 = 64$.

To derive the method of extracting the square root of a number, it is necessary to determine:

I. The relative number of places in a number and its square root.

II. The relations of the figures of the root to the periods of the number.

III. The law by which the parts of a number are combined in the formation of its square (§ 734).

750. The relative number of places in a given number and its square root is shown in the following illustrations:

Roots.	Squares.	Roots.	Squares.
1	1	1	1
9	81	10	1,00
99	98,01	100	1,00,00
999	99,80,01	1000	1,00,00,00

I. From these examples we perceive (1) that a root consisting of 1 place may have 1 or 2 places in the square.

(2) That the addition of 1 place to the root adds 2 places to the square.

751. Hence we have the following principles:

PRINCIPLES. — I. *If we point off a number into two-figure periods, commencing at the right hand, the number of full periods and the left-hand full or partial period will indicate the number of places in the square root, the highest period corresponding to the highest term of the root.*

II. To ascertain the relations of the several terms of the root to the periods of the number, observe that if any number, as 2345, is decomposed the squares of the left-hand parts will be related in local value, as follows:

$2000^2 = 4\ 00\ 00\ 00$ $\qquad$ $2340^2 = 5\ 47\ 56\ 00$

$2300^2 = 5\ 29\ 00\ 00$ $\qquad$ $2345^2 = 5\ 49\ 90\ 25$

II. *The square of the first term of the root is contained wholly in the first period of the power; the square of the first two terms of the root is contained wholly in the first two periods of the power; the square of the first three terms of the root is contained wholly in the first three periods of the power; and so on.*

NOTE. — *The periods and terms of the root are counted from the left hand.*

III. Since the square of a number expressed in tens and units $= t^2 + 2tu + u^2$ (§ 734), if we take away the square of the tens (t^2), the remainder will be $2tu + u^2$. Hence,

III. *If the square of the tens is subtracted from the entire square, the remainder will be equal to twice the product of the tens by the units, plus the square of the units.*

Examples.

752. To extract square root.

1. What is the length of one side of a square plot containing an area of 5417 sq. ft.?

Geometrical Explanation.

OPERATION.

	54,17 | 73.6
	49
140	517
143	429
146.0	88.00
146.6	87.96
	4

Solution. — Since the given figure is a square, its side will be the square root of its area, which we shall proceed to compute. Pointing off the given number, the two periods show that there will be two integral terms — tens and units — in the root. The tens of the root must be extracted from the first or left-hand period, 54 hundreds. The greatest square in 54 hundreds is 49 hundreds, the square of 7 tens; we therefore write 7 tens in the root, at the right of the given number.

Since the entire root is to be the side of a square, let us form a square (Fig I), the side of which is 70 feet long. The area of this square is $70 \times 70 = 4900$ sq. ft., which we subtract from the given number. This is done in the operation by subtracting the square number, 49, from the first period, 54, and to the remainder bringing down the second period, making the entire remainder 517.

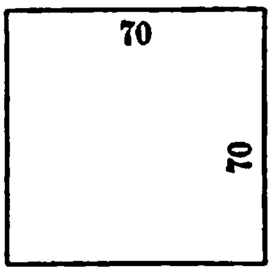


Fig. I.

If we now enlarge our square (Fig. I) by the addition of 517 square feet, in such a manner as to preserve the square form, its size will be that of the required square. To preserve the square form, the addition must be so made as to extend the square equally in two directions; it will therefore be composed of two oblong figures, at the sides and a little square at the corner (Fig. II). Now, the *width* of this addition will be the additional *length* to the side of the square, and consequently *the next figure in the root.* To find the *width*, we divide square contents, or area, by *length* (§ 117, I). But the length of one side of the little square cannot be found till the width of the addition is determined.

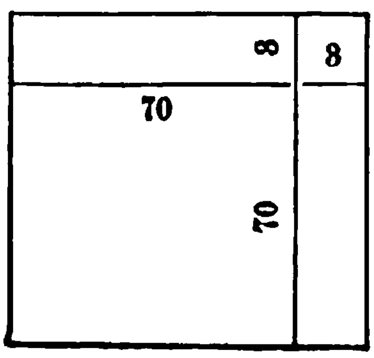


Fig. II.

because it is equal to this width. We therefore add the lengths of the two oblong figures, and the sum will be sufficiently near the whole length to be used as a trial divisor.

Each of the oblong figures is equal in length to the side of the square first formed; and their united length is 70 + 70 = 140 ft. (Fig. III). This number is obtained in the operation by doubling the 7 and annexing one cipher, the result being written at the left of the dividend. Dividing 517, the area, by 140, the approximate length, we obtain 3, the probable width of the addition, and the second figure of the root. Since 3 is also the side of the little square, we can now find the entire length of the addition, or the complete divisor, which is 70 + 70 + 3 = 143 (Fig. III). This number is found in the operation by adding 3 to the trial divisor, and writing the result underneath. Multiplying the complete divisor, 143, by the trial quotient figure 3, and subtracting the product from the dividend, we obtain another remainder of 88 square feet.

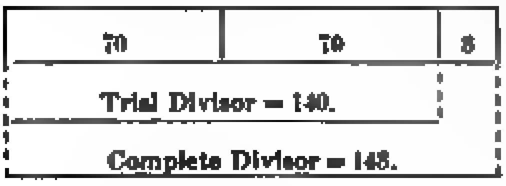


Fig. III.

With this remainder, for the same reason as before, we must proceed to make a new enlargement; and we bring down two decimal ciphers, because the next figure of the root being tenths, its square will be hundredths. The trial divisor to obtain the width of this new enlargement, or the next figure in the root, will be, for the same reason as before, twice 73, the root already found, with one cipher annexed. But since the 7 has already been doubled in the operation, we have only to double the last term of the complete divisor, 143, and annex a cipher, to obtain the new trial divisor, 146.0. Dividing, we obtain .6 for the trial term of the root; then proceeding as before, we obtain 146.6 for a complete divisor, 87.96 for a product; and there is still a remainder of .04. Hence, the side of the given square plot is 73.6 + feet.

Arithmetical Explanation.

We find the greatest square in 5400 as before, which is 4900, and place its root at the right. Since the square of a number divided into any two parts is equal to the square of the first part, plus twice the product of the first by the second, plus the square of the second part (§ 734), having found the square of the first part, which is 4900, the remainder 517 must be equal to twice the product of the first part 70 by the second part (?) plus the square of the second part. Since we do not know the second part, we take as a trial divisor twice the first, which is 140, and we find the second part to be about 3. Twice the product of the first by the second would be 3 times 140, and the square of the second, 3 × 3; or, since 3 × 140 + 3 × 3 = 3 × 143, we take 143 as a complete divisor and multiply it by the quotient figure 3, and obtain the product 429 and the remainder 88.

We now annex two decimal ciphers to the remainder because since the next figure of the root must be tenths its square must be hundredths. The trial divisor for this new enlargement will be, for the same reason as before, twice 73 with one cipher annexed, or 146.0. Dividing 88.00 by this, we find the quotient .6, the complete divisor, 146.6, the product 87.96 and a remainder of .04. Hence the root is 73.6 + feet.

RULE. — I. *Point off the given number into periods of two figures each, counting from units' place toward the left for whole numbers and toward the right for decimals.*

II. *Find the greatest square number in the left-hand period, and write its root for the first figure in the root; subtract this square number from the left-hand period, and to the remainder bring down the next period for a dividend.*

III. *At the left of the dividend write twice the first term of the root, and annex one cipher, for a trial divisor; divide the dividend by the trial divisor, and write the quotient for a trial figure in the root.*

IV. *Add the trial term of the root to the trial divisor for a complete divisor; multiply the complete divisor by the trial term in the root, and subtract the product from the dividend, and to the remainder bring down the next period for a new dividend.*

V. *To the last complete divisor add the last term of the root, and to the sum annex one cipher, for a new trial divisor, with which proceed as before.*

NOTES. — 1. If at any time the product is greater than the dividend, diminish the trial term of the root, and correct the erroneous work.

2. If a cipher occurs in the root, annex another cipher to the trial divisor, and another period to the dividend, and proceed as before.

3. If there is a remainder after all the periods have been brought down, annex periods of ciphers, and continue the root to as many decimal places as are required.

4. The decimal points in the work may be omitted, care being taken to point off in the root according to the number of decimal periods used.

5. The square root of a common fraction may be obtained by extracting the square roots of the numerator and denominator separately, provided the terms are perfect squares; otherwise, the fraction should first be reduced to a decimal.

6. Mixed numbers may be reduced to the decimal form before extracting the root; or, if the denominator of the fraction is a perfect square, to an improper fraction. The pupil will acquire greater facility, and secure greater accuracy, by keeping units of like order under each other, and each divisor opposite the corresponding dividend, as shown in the operation.

7. *The cipher in the trial divisor may be omitted, and its place, after division, may be occupied by the trial root term, thus forming only complete divisors.*

2. What is the square root of 406457.2516 ?

OPERATION.

		40,64,57.25,16 | 637.54 *Ans.*
		36
Trial divisor,	120	464
Complete "	123	369
Trial "	1260	9557
Complete "	1267	8869
Trial "	1274.0	688.25
Complete "	1274.5	637.25
Trial "	1275.00	51.0016
Complete "	1275.04	51.0016

3. What is the square root of 2 ?

	2.	| 1.41 + *Ans.*
	1	
	100	
24	96	
	400	
281	281	

Find the square root of:

4. 315844.
5. 152399025.
6. 56280004.
7. 5522500.
8. .013456
9. 10795.21.
10. $58.1406\frac{1}{4}$.
11. .0000316969.
12. 11916304.
13. 3486784401.
14. 29855296.
15. 5481918225.

Find the value of the following expressions:

16. $\sqrt{3}$.
17. $\sqrt{5\frac{5}{9}}$.
18. $\sqrt{10}$.
19. $\sqrt{3858.0769440964}$.
20. $\sqrt{99225 - 63504}$.
21. $\sqrt{75625 - 25000}$.
22. $\sqrt{.039177 - .025721}$.
23. $\sqrt{.12419504 - .005032}$.
24. $\sqrt{.126736} - \sqrt{.045369}$.
25. $\sqrt{\frac{169}{196}} \times \sqrt{\frac{7056}{9216}}$.
26. $\sqrt{81^2 \times 625^2 \times 2^4}$.
27. $\sqrt{\frac{625}{6561} \times \frac{7056}{9216}}$.

753. To abbreviate the extraction of square root.

1. Find the square root of 8, correct to 6 decimal places.

OPERATION.

```
         |2.828427+  Ans.
          8.00,00,00
          4
48        400
          384
562        1600
           1124
5648        47600
            45184
5656         2416*
             2262
566           154
              113
57             41
               40
```

SOLUTION. — Extracting the square root in the usual way until we have obtained the 4 places, 2.828, the corresponding remainder is 2416, and the next trial divisor, with the cipher omitted, is 5656. We now omit to bring down a period of ciphers to the remainder, thus contracting the dividend 2 places; and we contract the divisor an equal number of places by omitting to annex the trial figure of the root, and regarding the right-hand figure, 6, as a rejected or redundant figure. We now divide as in contracted division of decimals (§ 292), bringing down each divisor in its place, with one *redundant* figure increased by 1 when the *rejected* figure is 5 or more, and carrying the tens from the redundant figure in multiplication. We observe that the entire root, 2.828427+, *contains as many places as there are places in the periods used.*

RULE. — I. *If necessary, annex periods of ciphers to the given number, and assume as many terms as there are places required in the root; then proceed in the usual manner until all the assumed terms have been employed, omitting the remaining terms, if any.*

II. *Form the next trial divisor as usual, but omit to annex to it the trial term of the root, reject one term from the right to form each subsequent divisor, and in multiplying regard the right-hand term of each contracted divisor as redundant.*

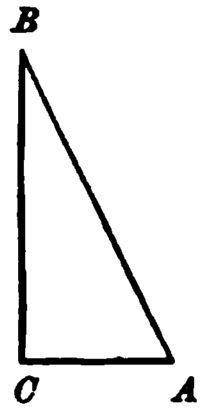
NOTES. — 1. If the rejected term is 5 or more, increase the next left-hand figure by 1.
2. Always take full periods, both of decimals and integers.

2. Find the sq. root of 32 correct to the 7th decimal place.
3. Find the sq. root of .5 correct to the 6th decimal place.
4. Find the sq. root of $6\frac{4}{7}$ correct to the 6th decimal place.
5. Find the sq. root of 1.06^5 correct to the 6th decimal place.
6. Find the value of $1.0125^{\frac{3}{2}}$ correct to the 4th decimal place.

Applications of Square Root.

754. An **Angle** is the opening between two lines that meet each other. Thus, the two lines AB and AC meeting, form an angle at A.

755. A **Right-angled Triangle** is a triangle having one right angle, as at C.



The **Base** is the side on which it stands, as the line AC in the diagram.

The **Perpendicular** is the side forming a right angle with the base, as the line BC.

The **Hypotenuse** is the side opposite the right angle, as the line AB.

756. **Similar Figures** are figures which have the same form and differ only in size.

757. The following principles, which are demonstrated in geometry, afford applications of square root:

Principles. — I. *The square of the hypotenuse of a right-angled triangle is equal to the sum of the squares of the other two sides; therefore,*

II. *The hypotenuse is equal to the square root of the sum of the squares of the other two sides.*

III. *The base or the perpendicular of a right-angled triangle is equal to the square root of the difference of the hypotenuse and that of the other side.*

IV. *The areas of two circles are to each other as the squares of their radii, diameters, or circumferences.*

V. *The ratio of the area of any two similar figures is equal to the square of the ratio of any two like dimensions of them; therefore,*

VI. *The ratio of any two like dimensions of any two similar figures is equal to the square root of the ratio of their areas.*

VII. *The mean proportional between two numbers is equal to the square root of their product* (§ 469, II).

Examples.

758. 1. The two sides of a right-angled triangle are 15 and 25 feet respectively. What is the length of the hypotenuse?

OPERATION.

$15^2 = 225$, square of one side.
$25^2 = \underline{625}$, square of the other side.
$ 850$, square of hypotenuse.
$\sqrt{850} = 29.15$ ft. *Ans.*

SOLUTION. — Squaring the two sides and adding, we find the sum to be 850 feet; and since the sum is equal to the square of the hypotenuse, we extract the square root, and obtain 29.15 feet, the hypotenuse.

2. A circular skating rink has a diameter of 75 feet. What would be the diameter of a similar rink with $\frac{1}{8}$ the area?

3. If an army of 55225 men is drawn up in the form of a square, how many men will there be on a side?

4. The diagonals of two similar rectangles are as 6 to 13. How many times does the larger contain the smaller?

5. The sides of two square blocks are 5 feet and 10 feet respectively. How do they compare in area?

6. A man has 200 yards of carpeting $1\frac{1}{8}$ yards wide. What is the length of one side of the square room which this carpet will cover?

7. How many rods of fence will be required to inclose 10 acres of land in the form of a square?

8. The end of a Maypole, broken 39 feet from the top, struck the ground 15 feet from the foot. What was the height of the pole?

9. A ladder 40 feet long is so placed in a street, that without being moved at the foot, it will reach a window on one side 33 feet, and on the other side 21 feet, from the ground. What is the breadth of the street?

10. Two men start from one corner of a park one mile square, and travel at the same rate. A goes by the walk around the park, and B takes the diagonal path to the opposite corner, and turns to meet A at the side. How many rods from the corner will the meeting take place?

11. The top of a castle is 45 yards high, and the castle is surrounded by a ditch 60 yards wide. What would be the length of a rope that would reach from the outside of the ditch to the top of the castle?

12. A ladder 52 feet long stands close against the side of a building. How many feet must it be drawn out at the bottom that the top may be lowered 4 feet?

13. A room is 20 feet long, 16 feet wide, and 12 feet high. What is the distance from one of the lower corners to the opposite upper corner?

14. It requires 63.39 rods of fence to inclose a circular field of 2 acres. What length will be required to inclose 3 acres in circular form?

15. The radius of a certain circle is 5 feet. What will be the radius of another circle containing twice the area of the first?

16. A certain circular race track has a diameter of 1500 feet. What would be the diameter of a similar track 4 times as large?

17. If it costs $ 167.70 to inclose a circular pond containing 17 A. 110 sq. rd., how much will it cost to inclose another $\frac{1}{3}$ as large?

18. If a cistern 6 feet in diameter holds 80 barrels of water, what is the diameter of a cistern of the same depth that holds 1200 barrels?

Find a mean proportional between:

19. 36 and 81. **21.** 64 and 12.25. **23.** $\frac{36}{49}$ and $\frac{4}{9}$.

20. 42 and 168. **22.** 8 and 288. **24.** $\frac{25}{64}$ and $\frac{169}{144}$.

25. The end of a pole broken 25 feet from the top struck the ground 15 feet from the foot. What was the height of the pole?

26. A tub of butter weighed 36 pounds by the grocer's scales, but being placed in the other scale of the balance it weighed only 30 pounds. What was the true weight of the butter?

CUBE ROOT.

759. The **Cube Root** of a number is one of the three equal factors that produce the number. Thus, the cube root of 343 is 7, since $7 \times 7 \times 7 = 343$.

To derive the method of extracting the cube root of a number, it is necessary to determine:

I. The relative number of places in a given number and its cube root.

II. The relations of the figures of the root to the periods of the number.

III. The law by which the parts of a number are combined in the formation of its cube (§ 735).

760. The relative number of places in a given number and its cube root is shown in the following illustrations:

Roots.	Cubes.	Roots.	Cubes.
1	1	1	1
9	729	10	1,000
99	907,299	100	1,000,000
999	997,002,999	1000	1,000,000,000

I. From these examples, we perceive: —

(1) That a root consisting of 1 place may have from 1 to 3 places in the cube.

(2) That, in all cases, the addition of 1 place to the root adds 3 places to the cube.

761. Hence we have the following principles:

PRINCIPLES. — I. *If we point off a number into three-figure periods, commencing at the right hand, the number of full periods and the left-hand full or partial period will indicate the number of places in the cube root.*

II. To ascertain the relations of the several figures of the root to the periods of the number, observe that if any number, as 5423, is decomposed, the cubes of the parts will be related in local value, as follows:

$$5000^3 = 125\ 000\ 000\ 000$$
$$5400^3 = 157\ 464\ 000\ 000$$
$$5420^3 = 159\ 220\ 088\ 000$$
$$5423^3 = 159\ 484\ 621\ 967.$$

II. *The cube of the first term of the root is contained wholly in the first period of the power; the cube of the first two terms of the root is contained wholly in the first two periods of the power; and so on.*

III. Since the cube of a number expressed in tens and units $= t^3 + 3t^2u + 3tu^2 + u^3$, if we take away the cube of the tens, t^3, the remainder will be $3t^2u + 3tu^2 + u^3$.

III. *If the cube of the tens is subtracted from the entire cube, the remainder will be three times the product of the tens squared by the units, plus three times the tens by the units squared, plus the cube of the units.*

Examples.

762. To extract cube root.

1. What is the length of one side of a cubical block containing 413494 solid inches?

GEOMETRICAL EXPLANATION.

OPERATION — COMMENCED.

$$\begin{array}{r|l} 413{,}494 & 74 \\ 343 & \\ \hline 14700 \quad 70494 & \end{array}$$

SOLUTION. — Since the block is a cube, its side will be the cube root of its solid contents, which we proceed to compute. Pointing off the given number, the two periods show that there will be two figures, tens and units, in the root. The tens of the root must be extracted from the first period, 413 thousands. The greatest cube in 413 thousands is 343 thousands, the cube of 7 tens; we therefore write 7 tens in the root at the right of the given number.

Since the entire root is to be the side of a cube, let us form a cubical block (Fig. I), the side of which is 70 inches in length. The contents of this cube are $70 \times 70 \times 70 = 343000$ solid inches, which we subtract from the given number. This is done in the operation by subtracting the cube number, 343, from the first period, 413, and to the remainder bringing down the second period, making the entire remainder 70494.

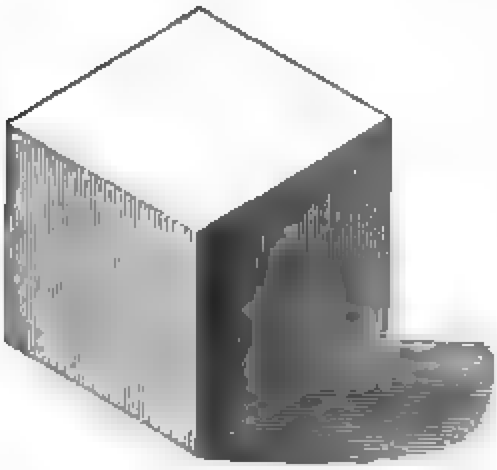


FIG. I.

If we now enlarge our cubical block (Fig. I) by the addition of 70494 solid inches, in such a manner as to preserve the cubical form, its size will be that of the required block. To preserve the cubical form, the addition must be made upon three adjacent sides or faces. The addition will therefore be composed of 3 flat blocks to cover the 3 faces (Fig. II); 3 oblong blocks to fill the vacancies at the edges (Fig. III); and 1 small cubical block to fill the vacancy at the corner (*Fig. IV*). Now, the *thickness* of this enlargement will be the addi-

tional length of the *side of the cube*, and, consequently, the *second figure in the root.* To find *thickness*, we may divide solid contents by *surface*, or *area* (§ 120). But the area of the 3 oblong blocks and little cube cannot be found till the thickness of the addition is determined, because their common breadth is equal to this thickness. We must therefore find the area of the 3 flat blocks (Fig. II), which is sufficiently near the whole area to be used as a *trial divisor*. As these are each equal in length and breadth to the side of the cube whose faces they cover, the whole area of the three is $70 \times 70 \times 3 = 14700$ square inches. This number is obtained in the operation by annexing 2 ciphers to 3 times the square of 7; the result being written at the left hand of the dividend. Dividing, we obtain 4, the probable thickness of the addition, and the second figure of the root. With this assumed term, we must *complete* our divisor by adding the area of the 4 blocks, before undetermined. The 3 oblong blocks are each 70 inches long; and the little cube, being equal in each of its dimensions to the thickness of the addition, must be 4 inches long. Hence, their united length is $70 + 70 + 70 + 4 = 214$. This number is obtained in the operation by multiplying the 7 by 3, and annexing the 4 to the product, the result being written in column I, on the next line below the trial divisor. Multiplying 214, the length, by 4, the common width, we obtain 856, the area of the four blocks, which added to 14700, the trial divisor, makes 15556, the *complete divisor;* and multiplying this by 4, the second figure in the root, and subtracting the product from the dividend, we obtain a remainder of 8270 solid inches. With this remainder, for the same reason as before, we must proceed to make a new enlargement. But since we have already two figures in the root, corresponding to the two periods of the given number, the next figure of the root must be a decimal; and we therefore annex to the remainder a period of three decimal ciphers, making 8270.000 for a new dividend.

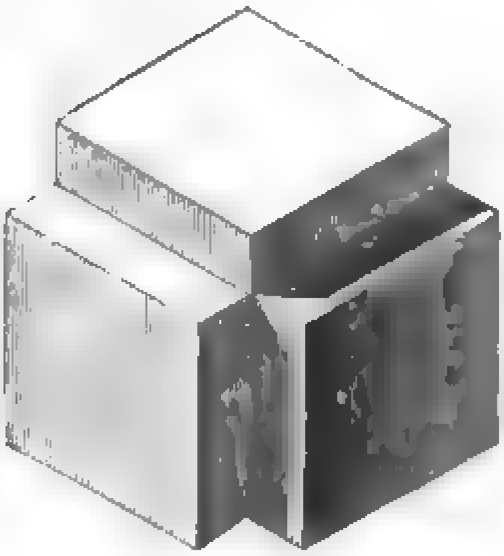


FIG. II.

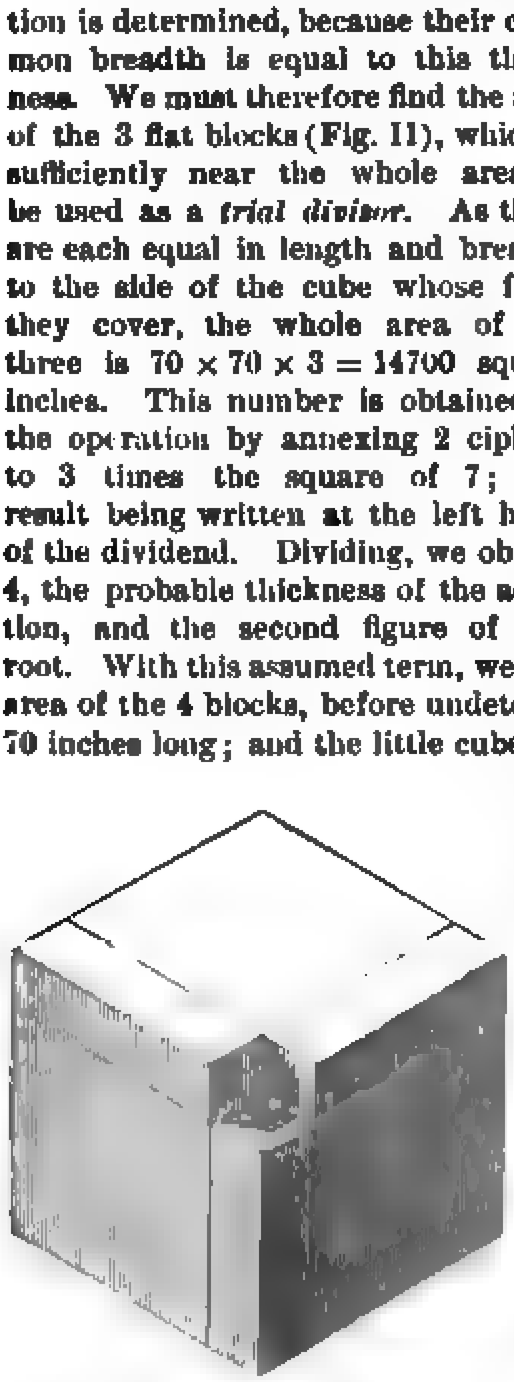


FIG. III.

[redacted]

The trial divisor to obtain the thickness of this second enlargement, or the next figure of the root, will be the area of three new flat blocks to cover the three sides of the cube already formed; and this surface to be covered (Fig. IV) is composed of 1 face of each of the flat blocks already used, 2 faces of each of the oblong blocks, and 3 faces of the little cube. But we have in the complete divisor, 15556, 1 face of each of the flat blocks, oblong blocks, and little cube; and in the correction of the trial divisor, 856, 1 face of each of the oblong blocks and of the little cube; and in the square of the last root figure, 16, a third face of the little cube. Hence, 16 + 856 + 15556 = 16428, the significant figures of the new trial divisor. This number is obtained in the operation by adding the square of the last root figure mentally, and combining units of like order, thus: 16, 6, and 6 are 28, and we write the unit figure in the new trial divisor; then 2 to carry, and 5 and 5 are 12, etc. We annex 2 ciphers to this trial divisor, as to the former. Or we may obtain this trial divisor, in the same way as the first one, by taking 3 times the square of 74 (which we regard as the first part), with one cipher annexed. Dividing, we obtain 5, the third figure in the root. To complete the second trial divisor, after the manner of the first, the correction may be found by annexing .5 to 3 times the former figures, 74, and multiplying this number by .5. But as

OPERATION — CONTINUED.

I.	II.			
			413,494	74
			343	
		14700	70494	
214	856	15556	62224	
			8270.000	

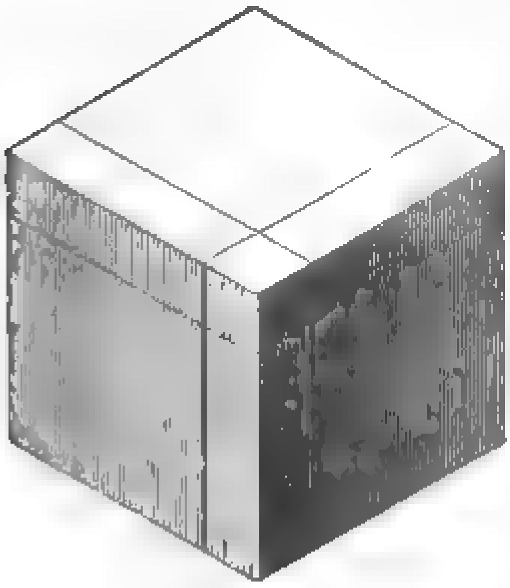


Fig. IV.

OPERATION — CONTINUED.

I.	II.			
			413,494	74.5
			343	
		14700	70494	
214	856	15556	62224	
		16428.00	8270.000	
222.5	111.25	16539.25	8269.625	
			.375	

we have, in column I, 3 times 7, with 4 annexed, or 214, we need only multiply the last figure, 4, by 3, and annex .5, making 222.5, which,

multiplied by .5, gives 111.25, the correction required. Then we obtain the complete divisor, 16539.25, the product, 8269.625, and the remainder, .375, in the manner shown by the former steps.

Arithmetical Explanation.

We find that the greatest cube in 413494 is 343000, whose cube root is 70, which we write at the right. Since the cube of a number divided into any two parts is equal to the cube of the first part, plus 3 times the square of the first by the second, plus 3 times the first by the square of the second, plus the cube of the second part (§ 735), therefore, having found the cube of the first part 70, which is 343000, the remainder 70494 must be equal to 3 times the square of the first part by the second part (?), plus 3 times the first part by the square of the second part, plus the cube of the second part. Since we do not know the second part, we take as a trial divisor 3 times the square of the first, which is 14700, and we find that the second part is about 4. The product of the quotient by the trial divisor will be 3 times the square of first by the second. To this must be added 3 times the first by the square of the second $= 3 \times 70 \times 4^2$ (but since the 4 in the quotient forms one of these factors we add to the trial divisor $3 \times 70 \times 4 = 840$), and the cube of the second, making the square 4×4 for addition to the trial divisor, and our complete divisor will be $14700 + 840 + 16 = 15556$, which multiplied by the quotient figure 4 gives a product of 62224. With the remainder, we proceed as before and the root is 74.5 +.

Rule.—I. *Point off the given number into periods of three figures each, counting from units' place toward the left for whole numbers and toward the right for decimals.*

II. *Find the greatest cube that does not exceed the left-hand period, and write its root for the highest term in the required root; subtract the cube from the left-hand period, and to the remainder bring down the next period for a dividend.*

III. *At the left of the dividend write three times the square of the first term of the root, and annex two ciphers, for a trial divisor; divide the dividend by the trial divisor, and write the quotient for a trial term in the root.*

IV. *Annex the trial term to three times the first term, and write the result in a column marked* I, *one line below the trial divisor; multiply this term by the trial term, and write the product on the same line in a column marked* II; *add this term as a correction to the trial divisor, and the result will be the complete divisor.*

V. *Multiply the complete divisor by the trial term, and subtract the product from the dividend, and to the remainder bring down the next period for a new dividend.*

VI. *Add the square of the last term of the root, the last term in column* II, *and the complete divisor together, and annex two ciphers, for a new trial divisor; with which obtain another trial term in the root.*

VII. *Multiply the units of the last term in column* I *by* 3, *and annex the trial term of the root for the next term of column* I; *multiply this result by the trial term of the root for the next term of column* II; *add this term to the trial divisor for a complete divisor, with which proceed as before.*

VIII. *If there is a remainder after the root of the last period is found, annex periods of ciphers and proceed as before. The figures of the root thus obtained will represent decimals.*

Note. — If at any time the product is greater than the dividend, diminish the trial term of the root, and correct the erroneous work. If a cipher occurs in the root, annex two more ciphers to the trial divisor, and another period to the dividend; then proceed as before with column I, annexing both cipher and trial term. The cube root of a fraction may be found by extracting the cube root of the numerator and denominator. In extracting the cube root of *decimal* numbers, begin at units' place and proceed toward the right, to separate into periods of *three* figures each.

1. What is the cube root of 79.112?

OPERATION.

I.	II.		79.112 | 4.2928 + *Ans.*
			64.
		4800	15112
122	244	5044	10088
		529200	5024000
1269	11421	540621	4865589
		55212300	158411000
12872	25744	55238044	110476088
		5526379200	47934912000
128768	1030144	5527409344	44219274752
			3715637248, Rem.

Find the cube root of:

2. 389017.
3. 44361864.
4. .091125.
5. 30.625.
6. $111\frac{1}{8}$.
7. .005.
8. 5.
9. 4.

Find the values of the following expressions:

10. $\sqrt[3]{122615327232}$.

11. $\sqrt[3]{\sqrt[3]{134217728}}$.

12. $\sqrt[3]{39304^2}$.

13. $\sqrt[3]{\frac{648}{3000}} \times \sqrt{\frac{1331}{3179}}$.

14. How much does the sum of the cube roots of 50 and 31 exceed the cube root of their sum?

763. To abbreviate the extraction of cube root.

In applying contracted decimal division to the extraction of the cube root of numbers, we observe:

(1) For each new figure in the root, the terms in the operation extend to the right 3 places in the column of dividends, 2 places in the column of divisors, and 1 place in column I. Hence,

(2) If at any point in the operation we omit to bring down new periods in the dividend, we must shorten each succeeding divisor 1 place, and each succeeding term in column I, 2 places.

1. What is the cube root of 189, correct to 8 decimal places?

OPERATION.

I.	II.		5.73879355 ± *Ans.*
			189.000,000
			125
		7500	64 000
157	1099	8599	60 193
		974700	3 807000
1713	5139	979839	2 939517
		984987	867483*
1719	1375	986362	789090
		98774	78393
17	12	98786	69150
		9880	9243
			8892
		988	351
			296
		99	55
			50
		10	5
			5

SOLUTION. — We proceed by the usual method to extract the cube root of the given number until we have obtained the three figures 5.73: the corresponding remainder is 867483, and the next trial divisor with the ciphers omitted is 984987. We now omit to bring down a period of ciphers, thus contracting the dividend 3 places; and we contract the divisor an equal number of places by omitting to annex the two ciphers, and regarding the right-hand figure, 7, as a redundant figure. Then dividing, we obtain 8 for the next figure of the root. To complete the divisor, we obtain a correction, 1375, contracted 2 places by omitting to annex the trial

term of the root, 8, to the first factor, 1719, and regarding the right-hand figure, 9, as redundant in multiplying. Adding the contraction to the contracted divisor, we have the complete divisor, 986362, the right-hand figure being redundant. Multiplying by 8 and subtracting the product from the dividend, we have 78393 for a new dividend. Then to form the new trial divisor, we disregard the square of the root term, 8, because this square consists of the same orders of units as the two rejected places in the divisor; and we simply add the correction, 1375, and the complete divisor, 986362, and rejecting 1 figure, thus obtain 98774, of which the right-hand figure, 4, is redundant. Dividing, we obtain 7 for the next root figure. Rejecting 2 places from the last term in column I, we have 17 for the next contracted term in this column. We then obtain, by the manner shown in the former step, the correction, 12, the complete divisor, 98786, the product, 69150, and the new dividend, 9243. We then obtain the new trial divisor, 9880; and as column I is *terminated* by rejecting the two places, 17, we continue the contracted division as in square root, and thus obtain the entire root, 5.73879355±, which is correct to the last decimal place, and *contains as many places as there are places in the periods used.*

RULE.—I. *If necessary, annex ciphers to the given number, and assume as many terms as there are places required in the root; then proceed by the usual method until the assumed terms have been employed.*

II. *Form the next trial divisor as usual, but omit to annex the two ciphers, and reject one place in forming each subsequent trial divisor.*

III. *In completing the contracted divisors, omit at first to annex the trial term of the root to the term in column* I, *and reject two places in forming each succeeding term in this column.*

IV. *In multiplying, regard the right-hand term of each contracted term, in column* I *and in the column of divisors, as redundant.*

NOTES.—1. After the contraction commences, the square of the last root term is disregarded in forming the new trial divisors.

2. Employ only *full periods* in the number.

Find the cube root of:

2. 24, correct to 7 decimal places.

3. 12000.812161, correct to 9 decimal places.

4. .171467, correct to 9 decimal places.

Find the value of:

5. $\sqrt[3]{\frac{5}{6}}$ to 6 places.

6. $\sqrt[3]{1.08674325^2}$ to 7 places.

7. $1.05^{\frac{2}{3}}$ to 7 places.

8. $\sqrt[3]{.57142\dot{8}}$ to 9 places.

APPLICATIONS OF CUBE ROOT.

764. The following principles of geometry afford applications of cube root.

PRINCIPLES.—I. *The ratio of two similar solids is equal to the cube of the ratio of any two like dimensions.*

II. *The ratio of any two like dimensions of similar solids is equal to the cube root of the ratios of the solids.*

Examples.

765. 1. The lengths of two similar solids are 4 inches and 50 inches. The first contains 16 cubic inches. What does the second contain?

2. If a ball 5 inches in diameter weighs 8 pounds, what will be the weight of a similar ball 10 inches in diameter?

3. I have two cubical boxes. The side of the smaller contains 6 inches, and of the larger, 10 inches. How many times would the smaller contain the larger?

4. What is the length of one side of a cistern of cubical form, containing 1331 solid feet?

5. The pedestal of a certain monument is a square block of granite, containing 373248 solid inches. What is the length of one of its sides?

6. A cubical box contains 474552 solid inches. What is the area of one of its sides?

7. A man wishes to make a bin to contain 125 bushels, of equal width and depth, and length double the width. What must be its dimensions?

8. If a cable 4 inches in circumference will support a sphere 2 feet in diameter, what is the diameter of that sphere which will be supported by a cable 5 inches in circumference?

ROOTS OF ANY DEGREE.

766. Any root whatever may be extracted by means of the square and cube roots, as will be seen in the two cases which follow.

Examples.

767. When the index of the required root contains no other factor than 2 or 3.

We have seen that if we raise any power of a given number to any required power, the result will be that power of the given number denoted by the product of the two indices (§ 732, III). Conversely, if we extract successively two or more roots of a given number, the result must be that root of the given number denoted by the product of the indices.

1. What is the 6th root of 2176782336?

OPERATION.

$$6 = 2 \times 3$$

$$\sqrt[2]{2176782336} = 46656$$

$$\sqrt[3]{46656} = 36 \quad \textit{Ans.}$$

Or,

$$\sqrt[3]{2176782336} = 1296$$

$$\sqrt[2]{1296} = 36 \quad \textit{Ans.}$$

SOLUTION. — The index of the required root is $6 = 2 \times 3$; we therefore extract the square root of the given number, and the cube root of this result, and obtain 36, which must be the 6th root required. Or, we first find the cube root of the given number, and then the square root of the result, as in the operation.

RULE. — *Separate the index of the required root into its prime factors, and extract successively the roots indicated by the several factors obtained; the final result will be the required root.*

2. What is the 6th root of 6321363049?
3. What is the 4th root of 5636405776?
4. What is the 8th root of 1099511627776?
5. What is the 6th root of 2563297285042049?
6. What is the 9th root of 1.577635?

NOTE. — Extract the cube root of the cube root by the contracted method, carrying the root in each operation to 6 decimal places only.

7. What is the 12th root of 16.3939?
8. What is the 18th root of 104.9617?

768. When the index of the required root is prime, or contains any other factor than 2 or 3.

To extract any root of a number is to separate the number into as many equal factors as there are units in the index of the required root; and it will be found that if by any means we can separate a number into factors nearly equal to each other, the *average* of these factors, or their sum divided by the number of factors, will be nearly equal to the root indicated by the number of factors.

1. What is the 7th root of 308?

OPERATION.

$$\sqrt[6]{308} = 2.59 +$$
$$\sqrt[8]{308} = 2.04 +$$
$$2.59 + 2.04 = 4.63$$
$$4.63 \div 2 = 2.31, \text{ assumed root.}$$

$$2.31^6 = 151.93$$
$$308 \div 151.93 = 2.0272 +$$
$$2.31 \times 6 + 2.0272 = 15.8872$$
$$15.8872 \div 7 = 2.2696, \text{ 1st approximation.}$$

$$2.2696^6 = 136.6748$$
$$308 \div 136.6748 = 2.253452 +$$
$$2.2696 \times 6 + 2.253452 = 15.871052$$
$$15.871052 \div 7 = 2.267293, \text{ 2d approximation.}$$

SOLUTION. — We first find the 6th root, and also the 8th root of 308; and since the 7th root must be less than the former and greater than the latter, we take the average of the two, or one half of their sums, 2.31, and call it the *assumed root.* We next raise the assumed root, 2.31, to the 6th power, and divide the given number, 308, by the result, and obtain 2.0272 + for a quotient; we thus separate 308 into 7 factors, 6 of which are equal to 2.31, and the other is 2.0272. As these 7 factors are nearly equal to each other, the average of them all must be a near approximation to the 7th root. Multiplying the 2.31 by 6, adding the 2.0272 to the product, and dividing this result by 7, we find the average to be 2.2696, which is the first approximation to the required root. We next divide 308 by the 6th power of 2.2696, and obtain 2.253452 + for a quotient; and we thus separate the given number into 7 factors, 6 of which are each equal to 2.2696, and the other is 2.253452. Finding the average of these factors, as in the former steps, we have 2.267293, which is the 7th root of the given number, correct to 5 decimal places.

[redacted]

Rule.—I. *Find by trial some number nearly equal to the required root, and call this the assumed root.*

II. *Divide the given number by that power of the assumed root denoted by the index of the required root less* 1; *to this quotient add as many times the assumed root as there are units in the index of the required root less* 1, *and divide the amount by the index of the required root. The result will be the first approximate root required.*

III. *Take the last approximation for the assumed root, with which proceed as with the former, and thus continue till the required root is obtained to what is considered a sufficient degree of exactness.*

Notes.—1. The involution and division in all cases will be much abridged by decimal contraction.

2. If the index of the required root contains the factors, 2 or 3, we may first extract the square or cube root as many times, successively, as these factors are found in the index, after which we must extract that root of the result which is denoted by the remaining factor of the index. Thus, if the 15th root were required, we should first find the cube root, then the 5th root of this result.

2. What is the 20th root of 617 ?

OPERATION.

$$20 = 2 \times 2 \times 5$$
$$\sqrt[2]{617} = 24.839485+$$
$$\sqrt[2]{24.839485} = 4.983923+$$
$$\sqrt[5]{4.983923} = 1.378206+ \quad \textit{Ans.}$$

3. What is the 5th root of 120 ?
4. What is the 7th root of 1.95678 ?
5. What is the 10th root of 743044 ?
6. What is the 15th root of 15 ?
7. What is the 25th root of 100 ?
8. What is the 5th root of 5 ?
9. What is the 5th root of 243 ?
10. What is the 5th root of 1024 ?
11. What is the 7th root of 16384 ?
12. What is the 10th root of 1048576 ?

PROGRESSIONS.

769. A **Progression** or **Series** is a succession of numbers that increase or decrease by a common law.

770. The **Terms** of a series are the numbers of which it is composed.

771. The **Extremes** are the first and last terms.

772. The **Means** are the intermediate terms.

773. An **Ascending Series** is one in which each term is greater than the one preceding, and a **Descending Series** is one in which each term is less than the one preceding. Thus, 3, 5, 7, 9, 11, etc., is an *ascending series,* and 13, 10, 7, 4, etc., is a *descending series.*

774. The **Law** of a series is the relation existing between the terms or the rate of increase or decrease of the terms.

ARITHMETICAL PROGRESSION.

775. An **Arithmetical Progression,** or **Series,** is a series of numbers increasing or decreasing by a constant common difference. Thus, the ascending and descending series given in § 773 are arithmetical progressions.

776. The **Common Difference** is the difference between any two adjacent terms.

777. There are *five parts* in an arithmetical series, any *three* of which being given, the other *two* may be found. They are as follows:

First term,
Last term,
Number of terms,
Common difference,
Sum of all the terms.

778. The conditions of a problem in progression may be such as to require any one of the five parts from any three of the four remaining parts; hence, in Arithmetical Progression there are $5 \times 4 = 20$ cases, or classes of problems, and no more, requiring each a different solution.

Examples.

779. To find one of the extremes when the other, the common difference, and number of terms are given.

1. Let 2 be the first term of an ascending series of 8 terms, and 3 the common difference. Find the last term. The series will be written 2, 5, 8, 11, 14, etc., or analyzed, thus, 2, $2+3$, $2+3+3$, $2+3+3+3$, $2+3+3+3+3$.

OPERATION.

$$7 \times 3 = 21$$

1st term, 2, $+21 = 23$, last term *Ans.*

SOLUTION. — Since the series will be 2, 5, 8, 11, 14, or analyzed 2, $2+3$, $2+3+3$, $2+3+3+3$, $2+3+3+3+3$, etc., we see that in an *ascending* series, we obtain the *second* term by *adding* the common difference *once* to the first term; the *third* term, by adding the common difference *twice* to the first term; the *fourth* term, by adding the common difference *three times* to the first term, etc.; and, in general, we obtain *any* term by adding the common difference *as many times* to the first term as there are terms less *one*. Hence, the last *term* will be equal to the first term, 2, plus $7 \times 3 = 23$ *Ans.*

RULE. — *Multiply the common difference by the number of terms less* 1, *and add the product to the first term, if the series is ascending, and subtract it if the series is descending.*

NOTE. — In other words, *to find the last term*, multiply the common difference by the number of terms less 1 and add the first term; *to find the first term*, multiply the common difference by the number of terms less 1 and subtract the last term.

2. The first term of an ascending series is 4, the common difference 3, and the number of terms 19. What is the last term?

3. What is the 13th term of a descending series whose first term is 75, and common difference 5?

4. The first term of an arithmetical progression is 5, the common difference 4, and the number of terms 8. What is the last term?

5. If the first term of an ascending series is 2, and the common difference 3, what is the 50th term?

[redacted]

6. The first term of a descending series is 100, the common difference 7, and the number of terms 13. What is the last term?

7. If the first term of an ascending series is $\frac{3}{8}$, the common difference $\frac{5}{8}$, and the number of terms 20, what is the last term?

8. A boy bought 18 hens, paying 2 cents for the first, 5 cents for the second, and 8 cents for the third, in arithmetical progression. How much did he pay for the last hen?

9. What is the 40th term of the series $\frac{1}{2}$, $\frac{3}{4}$, 1, $1\frac{1}{4}$, etc.?

10. A man travels 9 days; the first day he goes 20 miles, the second 25 miles, increasing his distance 5 miles each day. How far does he travel the last day of his journey?

11. What is the amount of $100, at 7%, for 45 years?

780. To find the common difference when the extremes and number of terms are given.

1. The first term is 2, the last term 23, and the number of terms 8. What is the common difference?

OPERATION.

Last term 23 − 1st term 2 = 21
$8-1=7$; $21 \div 7 = 3$, Com. Dif. *Ans.*

SOLUTION.—Referring to the series, 2, 5, 8, 11, 14, analyzed in § 779, we readily see that, by subtracting the *first term* from any term, we have left the *common difference* taken as many times as there are terms less 1. Thus, by taking away 2 in the fifth term, 2 + 3 + 3 + 3 + 3, we have left 12, which is 4 times 3. Hence, we divide the difference between 2 and 23, 21, by 8 − 1 = 7, and find the common difference 3.

RULE. — *Divide the difference of the extremes by the number of terms less* 1.

2. If the extremes of an arithmetical series are 3 and 15, and the number of terms 7, what is the common difference?

3. The extremes are 1 and 51, and the number of terms is 76. What is the common difference?

4. The first term is 2, the last term is 17, and the number of terms is 6. What is the common difference?

[redacted]

5. A man has seven children whose ages are in arithmetical progression; the youngest is 2 years old, and the eldest 14. What is the common difference of their ages?

6. The extremes of an arithmetical series are 1 and $50\frac{1}{2}$, and the number of terms is 34. What is the common difference?

7. An invalid commenced to walk for exercise, increasing the distance daily by a common difference; the first day he walked 3 miles, and the 14th day $9\frac{1}{2}$ miles. How many miles did he walk each day?

NOTE. — When we have found the common difference, we may add it once, twice, etc., to the first term, and we have the series, and consequently the *means*.

8. The first term of an arithmetical progression is 5, the last term 54, and the number of terms 8. What is the common difference?

9. The extremes are .05 and .1, and the number of terms is 8. What is the common difference?

10. If the extremes are 0 and $2\frac{1}{2}$, and the number of terms is 18, what is the common difference?

781. To find the number of terms when the extremes and common difference are given.

1. The extremes are 2 and 23, and the common difference 2. Find the number of terms.

OPERATION.

$23 - 2 = 21$

$21 \div 3 = 7$

$7 + 1 = 8$, No. terms *Ans.*

SOLUTION. — Examining the series, 2 5, 8, 11, 14, analyzed in § 779, we see that after taking away the *first* term from any term we have left the common difference taken as many times as the number of terms less 1. Hence we take away 2, the first term, from 23, the last term, and the remainder, 21, is 7 times 3. Since 7 is the number of terms less 1, the number is $7 + 1 = 8$.

RULE. — *Divide the difference of the extremes by the common difference, and add* 1 *to the quotient.*

2. The extremes are 5 and 75, and the common difference is 5. What is the number of terms?

3. The first term is $2\frac{1}{2}$, the last term is 40, and the common difference is $7\frac{1}{2}$. What is the number of terms?

4. A laborer agreed to build a fence on the following conditions: for the first rod he was to have 6 cents, with an increase of 4 cents on each successive rod; for the last rod he received 226 cents. How many rods did he build?

5. The extremes are $\frac{1}{2}$ and 20, and the common difference is $6\frac{1}{2}$. Find the number of terms.

782. To find the sum of all the terms when the extremes and number of terms are given.

1. The extremes are 2 and 14, and the number of terms 5. What is the sum of the series?

OPERATION.

$$\begin{array}{l} 2 + \ 5 + \ 8 + 11 + 14 = 40, \text{ once the sum.} \\ 14 + 11 + \ 8 + \ 5 + \ 2 = 40, \text{ " " "} \\ \hline 16 + 16 + 16 + 16 + 16 = 80, \text{ twice the sum.} \\ \qquad\qquad\qquad 80 \div 2 = 40, \text{ the sum } \textit{Ans.} \end{array}$$

SOLUTION. — To deduce a rule for finding the *sum of all the terms*, we will take the series 2, 5, 8, 11, 14, writing it under itself in an inverse order, and add each term.

Here we perceive that 16, the sum of the extremes, multiplied by 5, the number of terms, equals 80, which is *twice the sum of the series*. Dividing 80 by 2 gives 40, which is the sum required.

RULE. — *Multiply the sum of the extremes by the number of terms, and divide the product by* 2.

2. The extremes are 5 and 32, and the number of terms 12. What is the sum of all the terms?

3. How many strokes does a common clock make in 12 hours?

4. What debt can be discharged in a year by weekly payments in arithmetical progression, the first being \$24, and the last \$1224?

5. Suppose 100 apples were placed in a line 2 yards apart, and a basket 2 yards from the first apple. How far would a boy travel to gather them up singly, and return with each separately to the basket?

[redacted]

783. By reversing some one of the four problems now given, or by combining two or more of them, all of the sixteen remaining problems of Arithmetical Progression may be solved or analyzed.

1. The extremes are 0 and 250, and the number of terms is 1000. What is the sum of the series?

2. A person wishes to discharge a debt in 11 annual payments such that the last payment shall be $ 220, and each payment greater than the preceding by $ 17. Find the amount of the debt, and the first payment.

3. The first term of an arithmetical progression is 4, the common difference 5, and the number of terms 7. What is the sum of the series?

NOTE. — First find the last term by § 779, and then proceed as in § 782.

4. The extremes of an arithmetical progression are 8 and 64, and the common difference is 8. What is the sum of the series?

NOTE. — First find the number of terms by § 781, and then proceed as in § 782.

5. A man traveled 13 days; his last day's journey was 80 miles, and each day he traveled 5 miles more than on the preceding day. How far did he travel, and what was his first day's journey?

6. A bag of sand dropped from a balloon falls $16\frac{1}{12}$ ft. the first second and $32\frac{1}{6}$ ft. more each second than the one preceding. How far does it fall in 10 seconds?

7. The distance between two places is 360 miles. In how many days can it be traveled by a man who travels the first day 27 miles, and the last day 45, each day's journey being greater than the preceding by the same number of miles?

8. A farmer pays $ 1196 in 13 quarterly payments in such a way that each payment is greater than the preceding by $ 12. What are his first and last payments?

9. Find the first and last terms of an arithmetical progression whose sum is 408, common difference 6, and number *of terms 8.*

GEOMETRICAL PROGRESSION.

784. A **Geometrical Progression** is a series of numbers increasing or decreasing by a constant multiplier.

785. When the multiplier is *greater* than a unit, the series is **Ascending.** When the multiplier is *less* than a unit, the series is **Descending.**

Thus, 2, 6, 18, 54, 162, is an *ascending* series, in which 3 is the multiplier. 162, 54, 18, 6, 2, is a *descending* series, in which $\frac{1}{3}$ is the multiplier.

786. The **Ratio** is the common multiplier.

787. In every geometrical progression there are *five parts* to be considered, any *three* of which being given, the other *two* may be determined. They are as follows:

First term,	*Ratio,*
Last term,	*Sum of all the terms.*
Number of terms,	

788. Hence, as in Arithmetical Progression there are twenty different classes of problems, but the solution of all may be derived from the principles set forth in the following cases.

Examples.

789. To find one extreme, the other extreme, the ratio, and the number of terms being given.

1. The first term of a geometrical ratio is 2 and the multiplier or ratio is 3. What is the fourth term?

OPERATION.

$3^3 = 27$

$2 \times 27 = 54$, 4th term *Ans.*

SOLUTION.—The first term exists independently of the ratio. Since the number is multiplied by 3, the second term is 2×3, the third term $2 \times 3 \times 3$, or 2×3^2, the fourth term $2 \times 3 \times 3 \times 3$ or 2×3^3. Using the ratio *once* as a factor, gives the second term; using it *twice* or its *second power*, the *third* term; using it *three times* or its *third power*, the *fourth term.* The third power of 3 is 27, and the first term, 2, multiplied by 27 gives the fourth term, 54.

[redacted]

RULE. — I. To find the last term. — *Multiply the first term by that power of the ratio indicated by the number of terms less* 1.

II. To find the first term. — *Divide the last term by that power of the ratio indicated by the number of terms less* 1.

2. The first term of a geometrical series is 6, the ratio 4, and the number of terms 6. Find the last term.

3. The last term of a geometrical series is 192, the ratio 2, and the number of terms 7. What is the first term?

4. If the first term is 6, the ratio $\frac{1}{8}$, and the number of terms 8, what is the last term?

5. The first term is 25, the ratio $\frac{1}{5}$, and the number of terms 5. What is the last term?

6. A boy bought 9 oranges, agreeing to pay 1 cent for the first orange, 2 cents for the second, and so on. How much did the last orange cost him?

7. The first term is 7, the ratio $\frac{1}{7}$, and the number of terms 7. What is the last term?

8. What is the amount of $1 at compound interest for 5 years, at 7 % per annum?

NOTE. — The first term here is $1, the ratio is $1.07, and the number of terms is 6.

9. A drover bought 7 oxen, agreeing to pay $3 for the first ox, $9 for the second, $27 for the third, and so on. How much did the last ox cost him?

10. Find the 12th term of the series, 30, 15, $7\frac{1}{2}$, etc.

790. To find the ratio, the extremes and number of terms being given.

1. The first term of a geometrical progression is 2, the last term 54, and the number of terms 4. What is the ratio?

OPERATION.

$54 \div 2 = 27$

$\sqrt[3]{27} = 3$, ratio *Ans.*

SOLUTION. — Since the last term $54 = 2 \times$ the third power of the ratio (§ 789), $\sqrt[3]{54 \div 2}$ will be the ratio. Hence the ratio is 3.

RULE. — *Divide the last term by the first, and extract that root of the quotient indicated by the number of terms less* 1; *the result will be the ratio.*

2. The extremes are 2 and 512, and the number of terms is 5. What is the ratio?

3. The extremes are $\frac{1}{48}$ and $45\frac{9}{16}$, and the number of terms is 8. What is the ratio?

4. The extremes are 7 and .0112, and the number of terms is 5. What is the ratio?

5. Insert three geometrical means between 8 and 5000.

6. The first term of a geometrical progression is 1, the last term 15625, and the number of terms 7. Find the common ratio.

791. To find the number of terms, the extremes and ratio being given.

1. The first term of a geometrical progression is 2, the last term is 54, and the ratio 3. Find the number of terms.

OPERATION.

$54 \div 2 = 27$
$27 \div 3 = 9$
$9 \div 3 = 3$
$3 \div 3 = 1$
$3 + 1 = 4$ terms *Ans.*

SOLUTION.—If we represent the number of terms by n, $54 \div 2 = 3^{n-1}$ (§ 790). Hence $27 =$ a power of 3 indicated by the number of terms less 1. Dividing 27 by 3 until the quotient is 1, we find that 27 is the third power of 3. Hence $n - 1 = 3$ and n, the number of terms, $= 4$.

RULE.—*Divide the last term by the first, divide this quotient by the ratio, and the quotient thus obtained by the ratio again, and so on in successive division, till the final quotient is* 1. *The number of times the ratio is used as a divisor, plus* 1, *is the number of terms.*

2. The extremes are 2 and 1458, and the ratio is 3. What is the number of terms?

3. The first term is .1, the last term 100, and the ratio 10. Find the number of terms.

4. The first term is $\frac{1}{640}$, the last term $\frac{1}{5}$, and the ratio 2. What is the number of terms?

5. The extremes are 196608 and 6, and the ratio is $\frac{1}{8}$. What is the number of terms?

[redacted]

792. To find the sum of the series, the extremes and ratio being given.

1. Find the sum of a geometrical series in which the first term is 2, the last term 512, and the ratio 4.

OPERATION.

	$8 + 32 + 128 + 512 + 2048 = 2728 =$	*Four times* the sum of all the terms.
But	$2 + 8 + 32 + 128 + 512 \quad = \quad 682 =$	*Once* the sum of all the terms.
Hence, by subtracting, we get	$2048 - 2 = 2046 =$	*Three times* the sum of all the terms.
Dividing by 3, the ratio less 1,	$2046 \div 3 = \quad 682 =$	*Once* the sum of all the terms.

$(512 \times 4) - 2 = 2046$; $2046 \div 3 = 682$, sum *Ans.*

SOLUTION. — If we take the series 2, 8, 32, 128, 512, in which the ratio is 4, multiply each term by the ratio, and add the terms thus multiplied, we shall have the result shown in the operation.

The subtraction is performed by taking the *lower line* or series from the *upper*. All the terms cancel except 2048 and 2. Taking their difference, which is 3 times the sum, and dividing by 3, the ratio less 1, we must have the sum of all the terms.

RULE. — *Multiply the greater extreme by the ratio, subtract the less extreme from the product, and divide the remainder by the ratio less* 1.

NOTE. — Let every *decreasing* series be inverted, and the first term called the last; then the ratio will be *greater* than a unit. An *infinite series* is a descending series the number of whose terms is infinite. If the series is *infinite*, the first term is a cipher.

2. The extremes are 3 and 384, and the ratio is 2. What is the sum of the series?

3. If the extremes are 5 and 1080, and the ratio is 6, what is the sum of the series?

4. If the first term is $4\frac{4}{5}$, the last term $\frac{8}{405}$, and the ratio $\frac{1}{3}$, what is the sum of the series?

5. What is the sum of the infinite series, 8, 4, 2, 1, $\frac{1}{2}$, $\frac{1}{4}$, etc.?

6. The first term is 2, the last term 486, and the ratio 3. What is the sum of all the terms?

7. The first term is 4, the last term 262144, and the ratio 4. What is the sum of the series?

8. The first term of a descending series is 162, the last term 2, and the ratio $\frac{1}{3}$. What is the sum?

way that each payment is 3 times
ment is $10, and his last $7290.
debt?

13. A man wishes to discharge
making the first payment $2, the
ment 4 times the preceding paym
of his indebtedness?

14. Find the sum of 21 terms of

15. The first term of a geometric
term 512, and common multiplier 4.

**793. To find the sum of the series, t
the number of terms being given.**

1. The first term of a geometrical
3, and the number of terms 6. What

OPERATION.

$4 \times 3^5 =$ last

$$\frac{4 \times 3^5 \times 3 - 4}{3 - 1} = \frac{4 \times 3^6 - 4}{3 - 1} = \frac{4 \times (}{3}$$

SOLUTION. — We have all the conditions
term is needed. Since the first term, the r
are given, we find the last term by § 789.
§ 792, and is solved in the same way.

RULE. — *Raise the ratio to a power
terms, and subtract* 1 *from th*

[redacted]

4. The first term is 175, the ratio 1.06, and the number of terms 5. What is the sum of the series?

5. The first term is 4, the ratio 5, and the number of terms 5. What is the sum of the series?

6. What yearly debts can be discharged by monthly payments, the first being \$2, the second \$6, and the third \$18, and so on in geometrical progression?

7. Six persons of the Morse family came to this country 200 years ago. Suppose that their number has doubled every 20 years since, what would it be now?

794. To find the ratio, the extremes and the sum of the series being given.

1. The extremes of a geometrical progression are 2 and 162, and the sum of the series is 242. What is the ratio?

OPERATION.

$242 - 2 = 240$
$242 - 162 = 80$
$240 \div 80 = 3$, ratio *Ans.*

SOLUTION. — Let us assume in order to explain the process, that the ratio is 3. The series will then be $2 + 6 + 18 + 54 + 162$. If we remove the first term and the last term, successively, and compare the results,

$$6 + 18 + 54 + 162$$
$$2 + 6 + 18 + 54,$$

we find that each term in the first line is 3 times the corresponding term in the second line. Hence, the sum of the first line must be 3 times the sum of the second line; that is to say, the sum of the series minus the first term, 240, divided by the sum minus the last term, 80 = the ratio, 3.

RULE. — *Divide the sum of the series minus the first term, by the sum of the series minus the last term.*

2. The extremes are 2 and 686, and the sum of the series is 800. What is the ratio?

3. The extremes are $\frac{1}{4}$ and 64, and the sum of the series is $127\frac{3}{4}$. What is the ratio?

4. If the sum of an infinite series is $4\frac{1}{2}$, and the greater extreme 3, what is the ratio?

795. Every other problem in Geometrical Progression, that admits of an arithmetical solution, may be solved either by *reversing* or *combining* some of the problems already given.

Compound Interest by Geometrical Progression.

796. We have seen (§ 639) that if any sum at compound interest is multiplied by the amount of $ 1 for the given interval, the product will be the amount of the given sum or principal at the end of the first interval; and that this amount constitutes a new principal for the second interval, and so on for a third, fourth, or any other interval. Hence, a question in compound interest constitutes a geometrical progression:

First term	= Principal.
Ratio	= 1 + rate per cent for one interval.
Number of terms	= Number of intervals plus 1.
Last term	= Amount.

All the usual cases of compound interest and discount computed at compound interest can therefore be solved by the rules for geometrical progression.

Examples.

797. 1. Find the amount of $ 250 for 4 years, at 6% compound interest.

OPERATION.

$$\$ 250 \times 1.06^4 = \$ 250 \times 1.262477 = \$ 315.61925 \quad \textit{Ans.}$$

Solution. — Here we have $250 the first term, 1.06 the ratio, and 5 the number of terms, to find the last term. Then by § 789 we find the last term, which is the amount required.

2. What is the amount of $ 350 for 4 years, at 6 % per annum compound interest?

3. Of what principal is $ 150 the compound interest for 2 years, at 7 % ?

4. What sum at 6 % compound interest, will amount to $ 1000 in 3 years?

5. In how many years will $ 40 amount to $ 53.24, at 10 % compound interest?

6. At what rate per cent compound interest will any sum double itself in 8 years?

7. What is the present worth of $ 322.51, at 5 % compound interest, due 24 years hence?

ANNUITIES.

798. An **Annuity** is literally a sum of money which is payable annually. The term is, however, applied to a sum which is payable at any equal intervals, as monthly, quarterly, semi-annually, etc.

NOTE.—The term, interval, will be used to denote the time between payments.

Annuities are of three kinds: Certain, Contingent, and Perpetual.

799. A **Certain Annuity** is one whose period of continuance is definite or fixed.

800. A **Contingent Annuity** is one which begins or ends with a contingent event—as the birth or death of a person; and hence the period of its continuance is uncertain.

801. A **Perpetual Annuity** or **Perpetuity** is one which continues forever.

802. Each of these kinds is subject, in reference to its commencement, to the three following conditions:

1st. *It may be deferred, i.e.* it is not to be entered upon until after a certain period of time.

2d. *It may be reversionary, i.e.* it is not to be entered upon until after the death of a certain person, or the occurrence of some certain event.

3d. *It may be in possession, i.e.* it is to be entered upon at once.

803. An **Annuity in Arrears** or **Forborne** is one on which the payments were not made when due. Interest is to be reckoned on each payment of an annuity in arrears, from its maturity, the same as on any other debt.

804. The **Final Value** or **Amount** of an annuity is the sum of all the payments plus interest on each from the time it becomes due till the annuity ceases.

805. The **Present Value** of an annuity is the sum of money which at the given rate of interest will amount to the final value of the annuity.

806. The practical application of annuities includes leases, life estates, dower rights, rents, pensions, reversions, life insurance, salaries, etc. (See § 553, 6 and p. 327.)

Annuities at Simple Interest.

807. In reference to an annuity at simple interest, we observe:

1. The first payment becomes due at the end of the first interval, and hence will bear interest until the annuity is settled.
2. The second payment becomes due at the end of the second interval, and hence will bear interest for one interval less than the first payment.
3. The third payment will bear interest for one interval less than the second; and so on to any number of terms. Hence,
4. All the payments being settled at one time, each will be less than the preceding, by the interest on the annuity for one interval.

Thus if we have an annuity of $500 for 5 years, which remains unpaid, at the end of the first year the first payment, $500, will begin to draw interest for 4 years and will amount to $524; the second payment will draw interest for 3 years and will amount to $518; the third payment, for 2 years, will amount to $512; the fourth payment, for 1 year, will amount to $506; and the fifth payment, paid when due, will draw no interest and hence will be the same as the annuity, $500.

808. Hence we see that the amounts of the payments constitute a *descending arithmetical series*, which may be expressed as an *ascending series*, as follows:

First term	= Annuity.
Common difference	= *Interest* on annuity for one interval.
Number of terms	= Number of intervals between the commencement and settlement of the annuity.
Last term	= *Annuity plus its interest* for as many intervals less one as intervene between the commencement and settlement of the annuity.
Sum of all the terms	= Final Value or Amount of the Annuity.

The rules in Arithmetical Progression will therefore solve all problems in annuities at simple interest.

Examples.

809. 1. A man works for a farmer one year and six months, at $20 per month, payable monthly; and these wages remain *unpaid* until the expiration of the whole term of service. How

[redacted]

much is due to the workman, allowing simple interest at 6% per annum?

OPERATION.

$$\$\,20 + \$\,.10 \times 17 = \$\,21.70, \text{ 1st term.}$$

$$\frac{\$\,20 + \$\,21.70}{2} \times 18 = \$\,375.30, \text{ sum } \textit{Ans.}$$

SOLUTION. — Here the last month's wages, \$20, is the first term; the number of months, 18, is the number of terms; and the interest on 1 month's wages, \$.10, is the common difference. Then, by § 779, we find the last term, which is the amount of the first month's wages for 17 months; and by § 782 we find the sum of the series, which is the sum of all the wages and interest.

2. A father deposits annually for the benefit of his son, commencing with his tenth birthday, such a sum that on his 21st birthday the first deposit at simple interest amounts to \$ 210, and the sum due to his son to \$ 1860. How much is the deposit, and at what rate per cent is it deposited?

OPERATION.

$$\frac{\$\,1860 \times 2}{12} - \$\,210 = \$\,100, \text{ deposit.}$$

$$\frac{210 - 100}{11} = 10\%, \text{ rate } \textit{Ans.}$$

SOLUTION. — Here the \$210, the amount of the first deposit, is the last term; 12, the number of deposits, is the number of terms; and \$1860, the amount of all the deposits, and interests, is the sum of the series. By reversing § 782, we find the first term to be \$100, which is the annual deposit; and by § 780, we find the common difference to be \$10; hence $\frac{10}{100}$, or 10%, is the annual rate.

3. What is the amount of an annuity of \$ 150 for $5\frac{1}{2}$ years, payable quarterly, at $1\frac{1}{2}\%$ per quarter?

4. In what time will an annual pension of \$ 500 amount to \$ 3450, at 6% simple interest?

5. Find the rate per cent at which an annuity of \$ 6000 will amount to \$ 59760 in 8 years, at simple interest.

6. What is the present worth of an annuity of \$ 300 for 5 years, at 6%?

NOTE. — First find the amount by § 782, and then the present worth by § 658.

Annuities at Compound Interest.

810. An Annuity at compound interest constitutes a *descending geometrical progression*, which may be expressed as an *ascending geometrical progression* if we regard the terms as follows:

First term	= Annuity.
Ratio	= 1 + rate per cent for one interval, expressed decimally.
Number of terms	= Number of intervals.
Last term	= First term multiplied by 1 + the rate per cent for one interval raised to a power 1 less than the number of terms.

Sum of all the terms = Final Value or Amount of the Annuity.

811. To find the present value.—*First find the amount of the annuity at the given rate and for the given time, by* § 782; *then find the present value of this amount by dividing it by the amount of* $ 1 *at compound interest.*

812. The present value of a *reversionary annuity* is that principal which will amount, at the time the reversion expires, to what will then be the present value of the annuity.

813. The present value of a *perpetuity* is a sum whose interest equals the annuity.

814. Hence, it will be seen that all questions in Annuities at compound interest can be solved by the rules of Geometrical Progression, by substituting the corresponding terms.

Note. — Consult the compound interest table, pp. 370, 371, to find powers. 1.06^7 is the same as the amount of $ 1 for 7 years at 6 %.

Examples.

815. 1. What is the amount of an annuity of $ 500 which is 7 years in arrears, at 6% compound interest?

OPERATION.

$$\frac{\$\,500 \times (1.06^7 - 1)}{1.06 - 1} = \$\,251.815 \div .06 = \$\,4196.91\tfrac{2}{3} \quad \textit{Ans.}$$

Solution. — The payment now due, $ 500, is the first term of a geometrical ratio, 1.06, the amount of $ 1 for 1 year, is the ratio, and 7 the number of terms. Solving by § 782, we find the sum of the series, which is the amount of the annuity, to be $ 4196.91⅔.

[redacted]

2. What is the present worth of the annuity in Ex. 1?

OPERATION.

$$\$\,4196.91\tfrac{3}{8} \div 1.50363 = \$\,2791.18+ \quad \textit{Ans.}$$

SOLUTION. — The amount of the annuity is \$4196.91⅜. The amount of \$1 for 7 years at 6% is \$1.50363. Hence the present worth is \$4196.91⅜ ÷ 1.50363 = \$2791.18+.

3. Find the annuity whose amount for 5 years, at 6% compound interest, is \$2818.55.

OPERATION.

$$\frac{\$\,2818.55 \times 1.06 - 1}{1.06^5 - 1} = \$\,2818.55 \times \frac{.06}{.338226} = \frac{\$\,169.113}{.338226} = \$\,500 \quad \textit{Ans.}$$

SOLUTION. — \$2818.55 = sum of series; 5 = number of terms; 1.06 = ratio. Reversing the rule (§ 793), we find the answer to be \$500.

4. What is the present value of a reversionary lease of \$100 commencing 14 years hence and to continue 20 years, compound interest at 5%?

OPERATION.

$$\frac{\$\,100 \times (1.05^{20} - 1)}{1.05 - 1} = \frac{\$\,100 \times 1.653298}{.05} = \frac{\$\,165.3298}{.05}$$

$$= \$\,3306.596, \text{ Final Value.}$$

$$\$\,3306.596 \div 5.253348 = \$\,629.426, \text{ Pres. Worth. } \textit{Ans.}$$

SOLUTION. — We first find the value of the annuity in arrears for the 20 years, or its worth when it expires. \$100 is the first term, 20 the number of terms, and 1.05 the ratio, and we find the sum of the series or the final value, \$3306.596, by § 793. This is what the lease is worth, 20 + 14 = 34 years hence. Since the amount of \$1 for 34 years at 5% is \$5.253348, \$3306.596 ÷ 5.253348, or \$629.426, must be the present worth.

5. An annual pension of \$500 is in arrears 10 years. What is the amount now due, allowing 6% compound interest?

6. Allowing 6% compound interest on an annuity of \$200, which is in arrears 20 years, what is its present amount?

7. What is the present worth of an annuity of \$500 for 7 years, at 6% compound interest?

8. An annuity of \$200 for 12 years is in reversion 6 years. What is its present worth, compound interest at 6%?

9. Find the annuity whose amount for 25 years is *\$16459.35, at 6%* compound interest.

MENSURATION.*

816. **Geometry** treats of quantities which have extension and form. Such quantities are called **Magnitudes.**

817. **Mensuration** is the process of finding the number of units in extension by computation or measurement. It is the application of arithmetic to geometry.

818. **Extension** denotes that property of bodies, by virtue of which they occupy definite portions of space. Its dimensions are *length*, *breadth*, and *thickness.*

819. A **Point** is that which has position only.

820. **Direction** is relative position of points.

821. A **Line** has length, but neither breadth, nor thickness. A **Surface** has length and breadth, but no thickness. A **Solid** has length, breadth, and thickness.

LINES AND ANGLES.

822. A **Straight Line** is a line that does not change its direction. It is the shortest distance between two points.

823. A **Broken** or **Crooked Line** is one made up of two or more straight lines. It changes its direction at one or more points.

824. A **Curved Line** changes its direction at every point.

825. A **Uniform Curve** is one that changes its direction regularly.

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* Some of the problems under Mensuration have been explained before as applications of Multiplication, Denominate Numbers, Involution, Evolution, etc. They are here repeated.

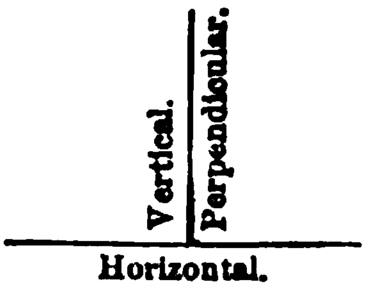
[redacted]

826. A **Varying Curve** is one that changes its direction irregularly.

827. **Parallel Lines** have the same direction; and being in the same plane and equally distant from each other, they can never meet.

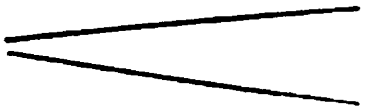
828. A **Horizontal Line** is a line parallel to the horizon or water level.

829. A **Perpendicular Line** is a straight line drawn to meet another straight line at right angles, that is, so as to incline no more to one side than to the other.



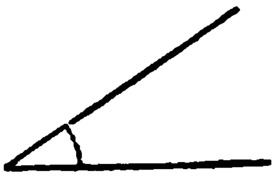
830. A perpendicular to a horizontal line is called a **Vertical Line.**

831. **Oblique Lines** approach each other, and will meet if sufficiently extended.



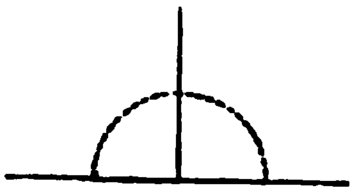
832. An **Angle** is the opening between two lines that meet each other in a common point, called the *vertex*.

833. Angles are measured by **Degrees** (§ 378).



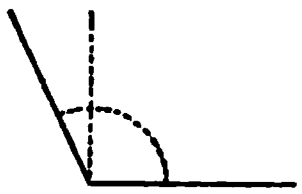
834. A **Right Angle** is an angle formed by two lines perpendicular to each other.

A right angle is always equal to 90°.



835. An **Obtuse Angle** is greater than a right angle.

It may be equal to any number of degrees more than 90° and less than 180°. At 180° the two lines forming the angle merge into one, and become a straight line.



836. An **Acute Angle** is less than a right angle.

An acute angle may be any number of degrees less than 90°.

837. *Obtuse* and acute angles are also called **Oblique Angles.**

PLANE FIGURES.

Polygons.

838. A **Plane Figure** is a portion of a plane surface bounded by straight or curved lines.

839. A **Polygon** is a plane figure bounded by straight lines.

840. The **Perimeter** of a polygon is the *sum* of its sides.

841. The **Center** of a regular polygon is the point within, equally distant from the middle points of the sides.

842. The **Altitude** of a polygon is the perpendicular distance from the highest point, or one of the highest points, to the line of the base.

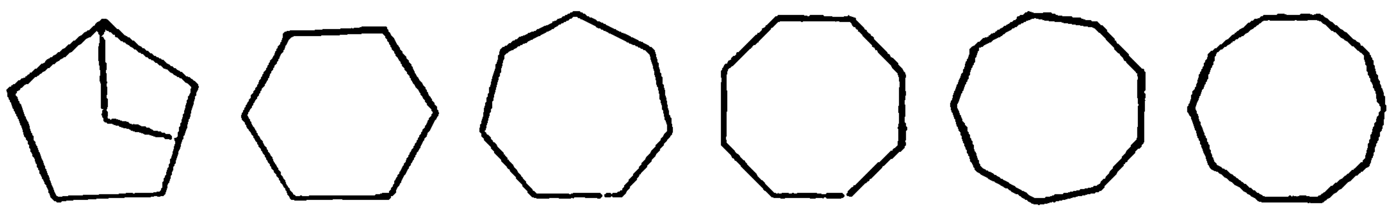
843. The **Apothem** of a regular polygon is the perpendicular line drawn from the center to the middle of a side.

844. The **Base** is the side on which a figure is supposed to stand.

845. The **Area** of a plane figure is the surface included within the lines which bound it.

A **Regular Polygon** has all its sides and all its angles equal.

A polygon of three sides is called a **Triangle**; of four sides, a **Tetragon**, or **Quadrilateral**; of five sides, a **Pentagon**, etc.



Pentagon. Hexagon. Heptagon. Octagon. Nonagon. Decagon.

Examples.

846. To find the area of a regular polygon.

1. Find the area of a regular hexagon, whose sides are each 3 in., and the apothem 1.732 in.

OPERATION.

$$\overline{6 \times 3} \times \overline{1.732 \div 2} = 15.588 \text{ sq. in. } \textit{Ans.}$$

Rule. — *Multiply the perimeter by half the apothem.*

Note. — Be careful in finding the product of two lines to express them in the same unit.

[redacted]

2. Find the area of a regular octagon, each of whose sides is 4 in. and its apothem 4.82 in.

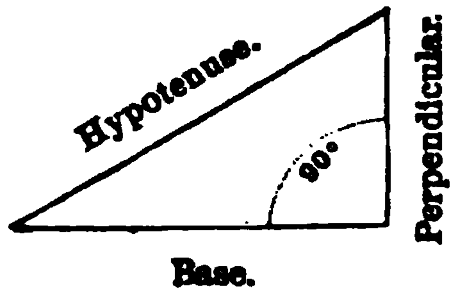
3. Find the area of a regular pentagon, whose side is $3\frac{1}{4}$ in. and its apothem 2.23 in.

4. Find the area of a regular decagon, whose side is 10 ft. and the apothem 15.38 ft.

Triangles.

847. A **Triangle** is a plane figure bounded by three sides, and having three angles.

848. A **Right-angled Triangle** is a triangle having one right angle.



849. The **Hypotenuse** of a right-angled triangle is the side opposite the right angle.

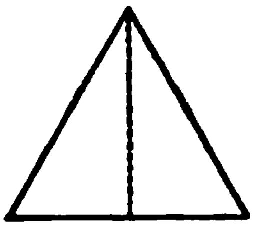
850. The **Base** of a triangle is the side on which it may be supposed to stand.

851. The **Perpendicular** of a right-angled triangle is the side which forms a right angle with the base.

852. The **Altitude** of a triangle is a line drawn perpendicular to the base from the angle opposite.

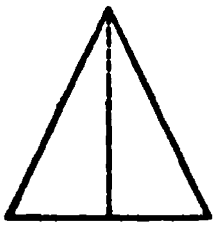
1. The dotted *vertical* lines in the figures represent the *altitude*.
2. Triangles are named from the relation both of their sides and angles.

Fig. 1.



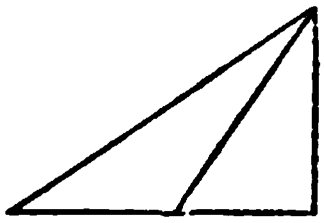
Equilateral.

Fig. 2.



Isosceles.

Fig. 3.



Scalene.

853. An **Equilateral Triangle** has its three sides equal.

854. An **Isosceles Triangle** has only two of its sides equal.

855. A **Scalene Triangle** has all of its sides unequal.

856. An **Equiangular Triangle** has three equal angles. (Fig. 1.)

857. An **Acute-angled Triangle** has three acute angles. (Fig. 2.)

858. An **Obtuse-angled Triangle** has one obtuse angle. (Fig. 3.)

Examples.

859. **The base and altitude of a triangle being given, to find its area.**

1. Find the area of a triangle whose base is 26 ft. and altitude 14.5 ft.

OPERATION.

$\overline{14.5 \times 26} \div 2 = 188\frac{1}{2}$ sq. ft. Or, $26 \times \frac{14.5}{2} = 188\frac{1}{2}$ sq. ft., area *Ans.*

RULE. — *Divide the product of the base and altitude by* 2. Or, *Multiply the base by one half the altitude.*

2. What is the area of a triangle whose altitude is 10 yd. and base 40 ft.?

Find the area of a triangle

3. Whose base is 12 ft. 6 in. and altitude 6 ft. 9 in.

4. Whose base is 25.01 ch. and altitude 18.14 ch.

5. What will be the cost of a triangular piece of land whose base is 15.48 ch. and altitude 9.67 ch., at $ 60 an acre?

6. At $ .40 a square yard, what is the cost of paving a triangular court, its base being 105 ft. and altitude 21 yd.?

7. Find the area of the gable end of a house that is 28 ft. wide, the ridge of the roof being 15 ft. higher than the foot of the rafters.

860. **The area and one dimension being given, to find the other dimension.**

1. What is the base of a triangle whose area is 189 sq. ft. and altitude 14 ft.?

OPERATION.

$\overline{189 \times 2} \div 14 = 27$ ft., base *Ans.*

RULE. — *Double the area, then divide by the given dimension.*

2. Find the altitude of a triangle whose area is $20\frac{1}{4}$ sq. ft. and base 3 yd.

Find the other dimension of the triangle

3. When the area is 65 sq. in. and the altitude 10 in.

4. When the base is 42 rd. and the area 588 sq. rd.

5. When the area is $6\frac{1}{4}$ acres and the altitude 17 yd.

6. When the base is 12.25 ch. and the area 5 A. 33 sq. rd.

7. I paid \$1050 for a piece of land in the form of a triangle, at the rate of \$$5\frac{1}{4}$ per square rod. If the base is 8 rd., what is its altitude?

861. The three sides of a triangle being given, to find its area.

1. Find the area of a triangle whose sides are 30, 40, and 50 ft.

OPERATION.

$$(30 + 40 + 50) \div 2 = 60;\ 60 - 30 = 30;$$
$$60 - 40 = 20;\ 60 - 50 = 10.$$
$$\sqrt{60 \times 30 \times 20 \times 10} = 600 \text{ ft., area } \textit{Ans.}$$

RULE.—*From half the sum of the three sides subtract each side separately; find the continued product of the half-sum and the three remainders; the square root of this product is the area.*

2. What is the area of an isosceles triangle whose base is 20 ft., and each of its equal sides 15 ft.?

3. How many acres are there in a field in the form of an equilateral triangle whose sides measure 70 rd.?

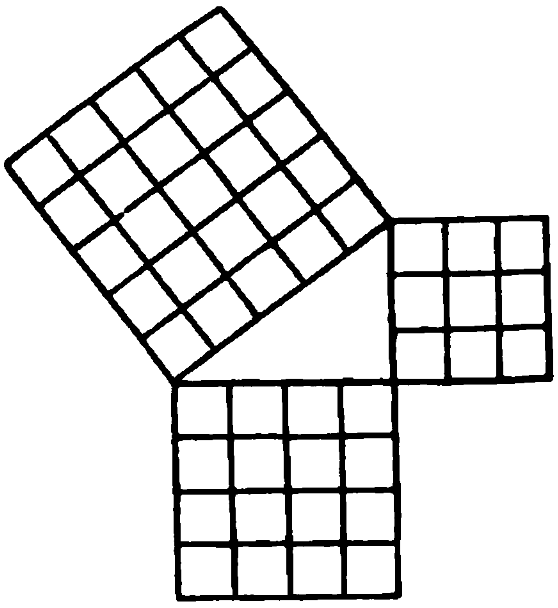
4. The roof of a house 30 ft. wide has the rafters on one side 20 ft. long, and on the other 18 ft. long. How many square feet of boards will be required to board up both gable ends?

5. The three sides of a triangular field are 56 rd. 72 rd. and 92 rd. respectively. What is its area?

6. Find the area of an isosceles triangle whose base is 16 in. and each of its sides 32 in.

862. The following principles relating to *right-angled* triangles have been established by geometry.

PRINCIPLES.—I. *The square of the hypotenuse of a right-angled triangle is equal to the sum of the squares of the other two sides.*



II. *The square of the base, or of the perpendicular, of a right-angled triangle is equal to the square of the hypotenuse diminished by the square of the other side.*

863. To find the hypotenuse.

1. The base of a right-angled triangle is 12, and the perpendicular 16. What is the length of the hypotenuse?

OPERATION.

$12^2 + 16^2 = 400$ (§ 862, I). $\sqrt{400} = 20$, hypotenuse *Ans.*

RULE.—*Extract the square root of the sum of the squares of the base and the perpendicular; the result will be the hypotenuse.*

2. If the gable end of a house 40 ft. wide is 16 ft. high, what is the length of the rafters?

3. A park 25 ch. long and 23 ch. wide has a walk running through it from opposite corners in a straight line. What is the length of the walk?

4. A room is 20 ft. long, 16 ft. wide, and 12 ft. high. What is the distance from one of the lower corners to the opposite upper corner?

864. To find the base or perpendicular.

1. The hypotenuse of a right-angled triangle is 35 ft., and the perpendicular is 28 ft. What is the base?

OPERATION.

$35^2 - 28^2 = 441$ (§ 862, II). $\sqrt{441} = 21$ ft., base *Ans.*

RULE.—*Extract the square root of the difference between the square of the hypotenuse and the square of the given side; the result will be the required side.*

[redacted]

2. The hypotenuse of a right-angled triangle is 53 yd. and the base is 84 ft. Find the perpendicular.

3. A line reaching from the top of a precipice 120 ft. high, on the bank of a river, to the opposite side is 380 ft. long. How wide is the river?

4. A ladder 39 ft. long stands against the side of a building. How many feet must it be drawn out at the bottom that the top may be lowered 3 ft.?

QUADRILATERALS.

865. A **Quadrilateral** is a plane figure bounded by four straight lines, and having four angles.

There are three kinds of quadrilaterals, — the **Parallelogram, Trapezoid,** and **Trapezium.**

866. A **Parallelogram** is a quadrilateral which has its opposite sides parallel.

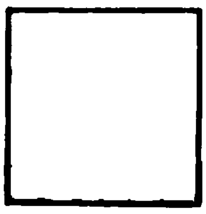
There are four kinds of parallelograms, — the **Square, Rectangle, Rhomboid,** and **Rhombus.**

867. A **Rectangle** is any parallelogram having all its angles right angles.

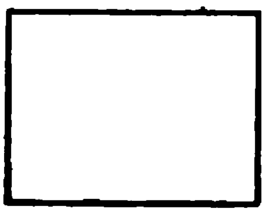
868. A **Square** is a rectangle whose sides are equal.

869. A **Rhomboid** is a parallelogram whose opposite sides *only* are equal, but whose angles are not right angles.

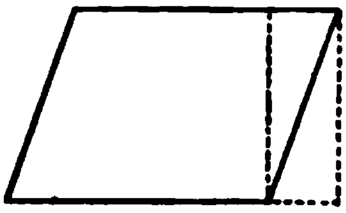
870. A **Rhombus** is a parallelogram whose sides are *all* equal, but whose angles are not right angles.



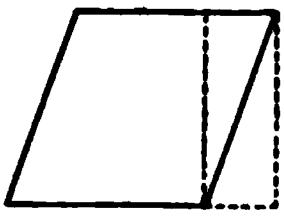
Square.



Rectangle.



Rhomboid.



Rhombus.

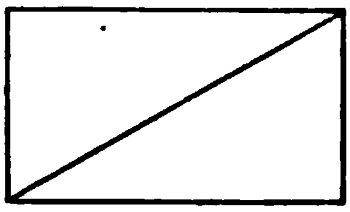
871. A **Trapezoid** is a quadrilateral, two of whose sides are *parallel* and two *oblique.*

872. A **Trapezium** is a quadrilateral having no two sides parallel.

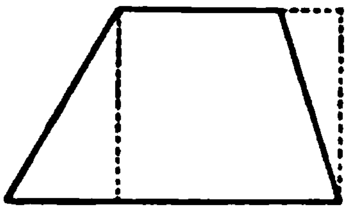
873. The **Altitude** of a parallelogram or trapezoid is the perpendicular distance between its *parallel* sides.

The *vertical* dotted lines in the figures represent the altitude.

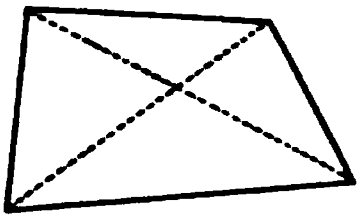
874. A **Diagonal** of a plane figure is a straight line joining the vertices of two angles not adjacent.



Parallelogram.



Trapezoid.



Trapezium.

Examples.

875. To find the area of any parallelogram.

1. Find the area of a parallelogram whose base is 16.25 ft. and altitude 7.5 ft.

OPERATION.

$$16.25 \times 7.5 = 121.875 \text{ sq. ft., area } \textit{Ans.}$$

RULE. — *Multiply the base by the altitude.*

2. The base of a rhombus is 10 ft. 6 in., and its altitude is 8 ft. What is its area?

3. How many acres are there in a piece of land in the form of a rhomboid, the base being 8.75 ch. and the altitude 6 ch.?

4. What is the area in square centimeters of a rectangle which is 41^{dm} long and 52^{dm} wide?

5. What is the width of a rectangle 4^{cm} long which contains $14^{sq\ cm}$?

876. To find the area of a trapezoid.

1. Find the area of a trapezoid whose parallel sides are 23 ft. and 11 ft., and the altitude 9 ft.

OPERATION.

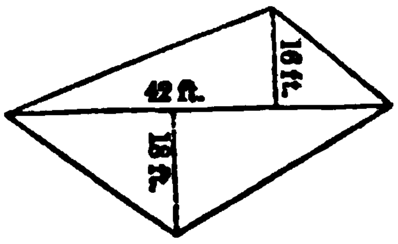
$$(\overline{23 + 11} \div 2) \times 9 = 153 \text{ sq. ft., area } \textit{Ans.}$$

RULE. — Multiply one *half the sum of the parallel sides by the altitude.*

2. What is the area of a trapezoid whose parallel sides are 178 ft. and 146 ft. and the altitude 69 ft.?

3. How many square feet are there in a board 16 ft. long, 18 in. wide at one end and 25 in. wide at the other end?

4. One side of a quadrilateral field measures 38 rd.; the side opposite and parallel to it measures 26 rd., and the distance between the two sides is 10 rd. Find the area.



877. To find the area of a trapezium.

1. Find the area of a trapezium whose diagonal is 42 ft., the perpendiculars to this diagonal, as in the diagram, being 16 ft. and 18 ft.

OPERATION.

$$(\overline{18 + 16 \div 2}) \times 42 = 714 \text{ sq. ft., area } \textit{Ans.}$$

RULE. —*Multiply the diagonal by half the sum of the perpendiculars drawn to it from the vertices of the opposite angles.*

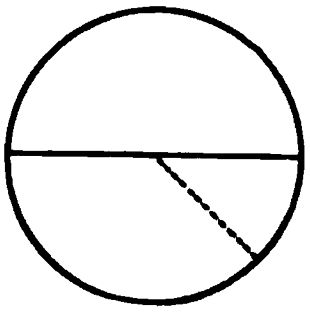
2. Find the area of a trapezium whose diagonal is 35 ft. 6 in., and the perpendiculars to this diagonal 9 ft. and 12 ft. 6 in.

3. How many acres are there in a quadrilateral field whose diagonal is 80 rd., and the perpendiculars to this diagonal 20.453 and 50.832 rd.?

To find the area of any *regular* polygon, multiply its perimeter, or the sum of its sides, by the perpendicular falling from its center to one of its sides.

To find the area of any *irregular* polygon, divide the figure into triangles and trapeziums, and find the area of each separately. The sum of these areas will be the area of the whole polygon.

CIRCLES.



878. A **Circle** is a plane figure bounded by a curved line, called the *circumference*, every point of which is equally distant from a point within called the *center*.

879. The **Diameter** of a circle is a line passing through its center, and terminated at both ends by the circumference.

880. The **Radius** of a circle is a line extending from its center to any point in the circumference. It is one half the diameter.

Examples.

881. When either the diameter or the circumference of a circle is given, to find the other dimension of it.

1. Find the circumference of a circle whose diameter is 20 in.

OPERATION.

20 in. × 3.1416 = 62.832 in. = 5 ft. 2.832 in., circum. *Ans.*

2. Find the diameter of a circle whose circumference is 62.832 ft.

OPERATION.

62.832 ft. ÷ 3.1416 = 20 ft., diameter *Ans.*

RULE. — I. *Multiply the diameter by* 3.1416; *the product will be the circumference.*

II. *Divide the circumference by* 3.1416; *the quotient will be the diameter.*

3. What is the circumference of a wheel 5 ft. 6 in. in diameter?

4. Find the diameter of a wheel whose circumference is 50 ft.

5. What is the diameter of a tree whose girth is 18 ft. 6 in.?

6. Find the length of tire that will band a wheel 7 ft. 9 in. in diameter.

7. The diameter of a cylinder is 8 ft. 6 in. Find its girth.

8. What is the radius of a circle whose circumference is 31.416 ft.?

9. The distance around the earth at the equator is 24899 miles. What is the length of the earth's equatorial diameter?

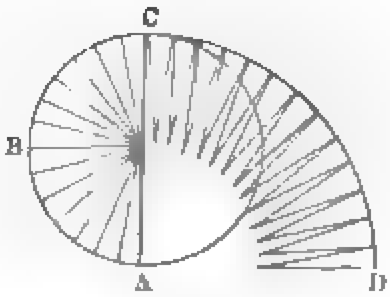
10. The radius of a circle is 10 ft. What is its circumference?

11. Find the circumference of the greatest circle that can be drawn with a string 14 in. long, used as a radius.

[redacted]

882. To find the area of a circle, when both its diameter and circumference are given, or when either is given.

NOTE. — A circle may be regarded as being composed of an indefinite number of triangles, the sum of whose bases form the circumference of the circle and whose altitude is the radius. Hence its area equals the circumference by ½ the radius or ¼ the diameter.



1. Find the area of a circle whose diameter is 10 ft. and circumference 31.416 ft.

OPERATION.

$$31.416 \text{ ft.} \times \overline{10 \div 4} = 78.54 \text{ sq. ft., area } \textit{Ans.}$$

2. Find the area of a circle whose diameter is 10 ft.

OPERATION.

$$(10 \text{ ft.})^2 \times .7854 = 78.54 \text{ sq. ft., area } \textit{Ans.}$$

3. Find the area of a circle whose circumference is 31.416 ft.

OPERATION.

$$31.416 \text{ ft.} \div 3.1416 = 10 \text{ ft., diameter};$$
$$(10 \text{ ft.})^2 \times .7854 = 78.54 \text{ sq. ft., area } \textit{Ans.}$$

RULE. — I. *Multiply the square of the diameter by .7854.* Or, *Multiply ¼ of the diameter by the circumference.*

4. What is the area of a circular pond whose circumference is 200 chains?

5. The distance around a circular park is 1¼ miles. How many acres does it contain?

6. Find the area of a circle whose diameter is 15^{cm}.

7. Find the area of the largest circle that can be drawn by using as a radius a string 20 in. long.

883. To find the diameter or circumference of a circle, when the area is given.

1. What is the diameter of a circle whose area is 1319.472 sq. ft.?

OPERATION.

$$\sqrt{1319.472 \text{ sq. ft.} \div .7854} = 40.987 + \text{ ft. diam. } \textit{Ans.}$$

2. What is the circumference of a circle whose area is 19.635 sq. rd.?

OPERATION.

$\sqrt{19.635 \text{ sq. rd.} \div 3.1416} = 2.5$ rd., radius;
$2.5 \text{ rd.} \times 2 \times 3.1416 = 15.708$ rd., circumference *Ans.*

RULE. — I. *Divide the area by* .7854 *and extract the square root of the quotient ; the result will be the diameter.* Or,

Divide the area by 3.1416 *and extract the square root of the quotient ; the result will be the radius. The circumference is obtained by* § 881, I. Or,

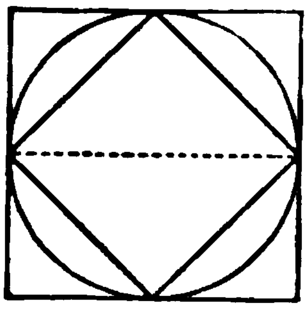
Divide the area by .07958, *and extract the square root of the quotient ; the result will be the circumference.*

3. The area of a circular lot is 38.4846 sq. rd. What is its diameter?

4. The area of a circle is 286.488 sq. ft. What are the diameter and the circumference?

5. The area of a circular lot is 1 acre. What is its diameter?

884. To find the side of an inscribed square when the diameter of the circle is known.


Inscribed and circumscribed square.

1. What is the side of a square inscribed in a circle whose diameter is 6 rods?

OPERATION.

$6^2 \div 2 = 18$; $\sqrt{18} = 4.24$ rd., side of sq. *Ans.*

RULE. — *Extract the square root of half the square of the diameter.* Or,

Multiply the diameter by .7071.

It is evident from the diagram that the side of a *circumscribed* square is the same length as the diameter of the circle about which it is circumscribed. Hence the area of a circumscribed square equals the square of the diameter, and since the area of the circle equals .7854 times the square of the diameter (§ 882) it follows that the area of the circle *equals* .7854 *times the area of the* circumscribed square.

2. The diameter of a circle is 200 ft. Find the side of the inscribed square.

3. The circumference of a circle is 104 yd. Find the side of the inscribed square.

4. What is the area of a square circumscribed about a circle whose diameter is 8 ft.?

5. What is the area of a square inscribed within a circle whose diameter is 2 ft.?

6. What is the area of a circle whose circumscribed square is 35 ft. long?

7. What is the length of a square circumscribed about a circle whose area equals 19.635 sq. yd.?

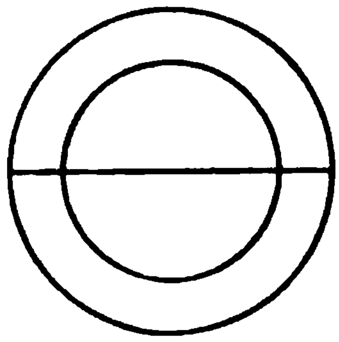
8. What is the diameter of a circle if the area of the inscribed square equals 512 sq. ft.?

9. What is the circumference of a circle if the area of the circumscribed square equals 9604 sq. rd.?

10. At $ .50 a rod, what is the cost of building a square fence about a circular pond whose diameter equals 35 rd.?

885. To find the area of a circular ring, formed by two concentric circles.

1. Find the area of a circular ring, when the diameters of the circles are 20 ft. and 30 ft.



OPERATION.

$$(30^2 \times .7854) - (20^2 \times .7854) = (30^2 - 20^2) \times .7854 = 500 \times .7854$$
$$= 392.7 \text{ sq. ft., area } Ans.$$

RULE.—*Multiply the difference between the squares of the diameters by .7854.*

2. Find the area of a circular ring formed by two concentric circles, whose diameters are 7 ft. 9 in. and 4 ft. 3 in.

3. The diameters of two concentric circles are 35.75 and 16.25 ft. Find the area of the ring.

4. The area of a circle is 1 A. 154.16 sq. rd. A pond in the center is 10 rd. in diam. Find the area of the land and of the *water.*

886. SUMMARY OF CIRCLES.

1. The *diameter* of any circle

$$\left.\begin{array}{l}\text{Multiplied}\\ \text{Divided}\end{array}\right\}\text{ by }\left\{\begin{array}{r}3.1416,\text{ the product}\\ .3183,\text{ the quotient}\end{array}\right\}=\text{the } \textit{circumference.}$$

$$\left.\begin{array}{l}\text{Multiplied}\\ \text{Divided}\end{array}\right\}\text{ by }\left\{\begin{array}{r}.8862,\text{ the product}\\ 1.1284,\text{ the quotient}\end{array}\right\}=\text{the } \textit{side of an equal square.}$$

$$\left.\begin{array}{l}\text{Multiplied}\\ \text{Divided}\end{array}\right\}\text{ by }\left\{\begin{array}{r}.8660,\text{ the product}\\ .1547,\text{ the quotient}\end{array}\right\}=\text{the } \textit{side of an inscribed equilateral triangle.}$$

$$\left.\begin{array}{l}\text{Multiplied}\\ \text{Divided}\end{array}\right\}\text{ by }\left\{\begin{array}{r}.7070,\text{ the product}\\ 1.4142,\text{ the quotient}\end{array}\right\}=\text{the } \textit{side of an inscribed square.}$$

2. The *radius* of any circle

$$\left.\begin{array}{l}\text{Multiplied}\\ \text{Divided}\end{array}\right\}\text{ by }\left\{\begin{array}{r}6.28318\\ .15915\end{array}\right\}=\text{the } \textit{circumference.}$$

3. The *square* of the diameter of any circle

$$\left.\begin{array}{l}\text{Multiplied}\\ \text{Divided}\end{array}\right\}\text{ by }\left\{\begin{array}{r}.7854,\text{ the product}\\ 1.2732,\text{ the quotient}\end{array}\right\}=\text{the } \textit{area.}$$

4. The *circumference* of any circle

$$\left.\begin{array}{l}\text{Multiplied}\\ \text{Divided}\end{array}\right\}\text{ by }\left\{\begin{array}{r}.3183,\text{ the product}\\ 3.1416,\text{ the quotient}\end{array}\right\}=\text{the } \textit{diameter.}$$

$$\left.\begin{array}{l}\text{Multiplied}\\ \text{Divided}\end{array}\right\}\text{ by }\left\{\begin{array}{r}.2821,\text{ the product}\\ 3.5450,\text{ the quotient}\end{array}\right\}=\text{the } \textit{side of an equal square.}$$

$$\left.\begin{array}{l}\text{Multiplied}\\ \text{Divided}\end{array}\right\}\text{ by }\left\{\begin{array}{r}.2756,\text{ the product}\\ 3.6276,\text{ the quotient}\end{array}\right\}=\text{the } \textit{side of an inscribed equilateral triangle.}$$

$$\left.\begin{array}{l}\text{Multiplied}\\ \text{Divided}\end{array}\right\}\text{ by }\left\{\begin{array}{r}.2251,\text{ the product}\\ 4.4428,\text{ the quotient}\end{array}\right\}=\text{the } \textit{side of an inscribed square.}$$

5. The *square* of the circumference of any circle

$$\left.\begin{array}{l}\text{Multiplied}\\ \text{Divided}\end{array}\right\}\text{ by }\left\{\begin{array}{r}.07958,\text{ the product}\\ 12.5663,\text{ the quotient}\end{array}\right\}=\text{the } \textit{area.}$$

6. The *area* of any circle

$$\left.\begin{array}{l}\text{Multiplied}\\ \text{Divided}\end{array}\right\}\text{ by }\left\{\begin{array}{r}1.2732,\text{ the product}\\ .7854,\text{ the quotient}\end{array}\right\}=\text{the } \textit{square of the diam.}$$

7. $$\left\{\begin{array}{l}\text{The square of the radius of any circle} \times 3.1416\\ \text{Half the circumference of a circle} \times \tfrac{1}{2}\text{ its diameter}\\ \text{Square of the circumference of a circle} \times .07958\end{array}\right\}=\textit{area.}$$

[redacted]

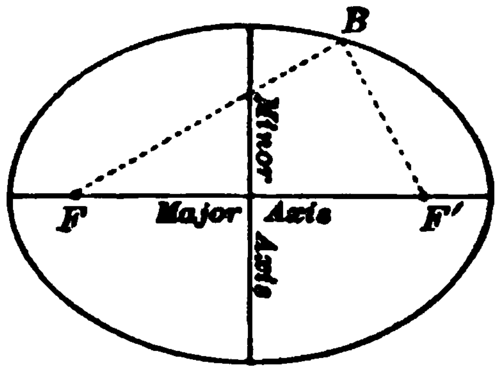
Ellipses.

887. An **Ellipse** is a plane figure bounded by a curved line called the **Periphery**, such that the sum of the distances of each point on the periphery from two fixed points within the **Foci** is the same.

An ellipse is the shadow of a circle.

888. A **Diameter** of an ellipse is a line passing through its center and terminated at both ends by the circumference.

889. The **Major Axis** is the greatest diameter of an ellipse and the **Minor Axis** is the least diameter.



Ellipse.

In the illustration, FF' are the foci, and the broken line FBF' or $FB+BF'$ represents the sum of the distances of B from the two foci, which is equal to the length of the major axis.

Examples.

890. To find the area of an ellipse.

1. What is the area of an ellipse whose major axis is 4 in. and the minor axis 2 in.?

Solution. — $4 \times 2 \times .7854 = 6.2832$ sq. in. *Ans.*

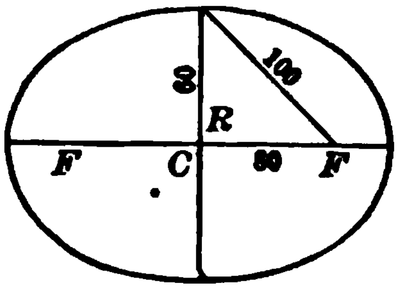
Rule. — *Multiply the product of the two axes by* .7854.

2. Find the area of an ellipse 6 in. $\times$ 15 in.

3. Find the area of an ellipse 3 in. $\times$ 24 in.; 8 ft. $\times$ 17 ft.

891. To find the minor axis.

1. What is the minor axis of an ellipse, whose major axis is 200 ft. and the distance between the foci 160 ft.?



Solution. — A line drawn from either focus to the extremity of the minor axis will be $\frac{1}{2}$ the major axis, and will form the hypotenuse of a right-angled triangle. Hence, $R = \sqrt{100^2 - 80^2} = 60$ and the minor axis is $2 \times 60 = 120$ ft. Ans.

RULE.—*Find the difference between $\frac{1}{2}$ the major axis, squared, and $\frac{1}{2}$ the distance between the foci, squared; extract the square root and multiply the result by* 2.

2. What is the minor axis of an ellipse, whose major axis is 30 ft. and the distance between the foci 10 ft.?

3. A rope 25 ft. long is passed through a ring to which a goat is tied, and the ends of the rope are fastened to two stakes 20 ft. apart, which are relatively north and south. How far east and west can the goat move, and over what part of a lawn whose dimensions are 75 ft. by 78.54 ft.?

892. To find the major axis.

1. What is the major axis of an ellipse, whose minor axis is 120 ft. and the distance between the foci 160 ft.?

SOLUTION.—Referring to the diagram in § 891, we have $R = 120 \div 2 =$ 60 ft. and $CF = 160 \div 2 = 80$ ft. Hence, the hypotenuse $= \sqrt{60^2 + 80^2} =$ 100 ft. This hypotenuse represents $\frac{1}{2}$ the major axis, and 2×100 ft. $= 200$ ft. *Ans.*

RULE.—*Find the sum of $\frac{1}{2}$ the minor axis, squared, and $\frac{1}{2}$ the distance between the foci, squared; extract the square root and multiply the result by* 2.

2. What is the major axis of an ellipse, whose minor axis is 30 ft. and the distance between the foci 60 ft.?

3. If the minor axis of an ellipse is 15 ft. and the distance between the foci is 20 ft., what is the area of the ellipse?

893. To find the difference between the foci.

1. If the major axis of an ellipse is 200 ft. and the minor axis 120 ft., what is the distance between the foci?

SOLUTION.—Referring to the diagram, $R = 60$ ft., and the hypotenuse being half the major axis $= 100$ ft.; we wish to find twice the other side, CF. Hence $\sqrt{100^2 - 60^2} \times 2 = 160$ ft., the distance between the foci *Ans.*

RULE.—*Find the difference between $\frac{1}{2}$ the major axis, squared, and $\frac{1}{2}$ the minor axis, squared; extract the square root and multiply the result by* 2.

2. How far apart are the foci of an ellipse whose major axis is 25 ft. long and the minor axis 15 ft.?

3. If the area of an ellipse is 294.525 sq. ft. and the major axis is 25 ft., what is the minor axis? What is the distance between the foci?

4. An ellipse whose minor axis is 30 in. has an area of 942.48 sq. in. How far apart are the foci?

Similar Plane Figures.

894. Similar Plane Figures are such as have the same *form;* or have angles equal each to each, the same number of sides, and the sides containing the equal angles proportional.

All circles, squares, equiangular triangles, and regular polygons of the same number of sides are similar figures.

The like dimensions of circles, that is, their radii, diameters, and circumferences, are proportional.

PRINCIPLES. — I. *The like dimensions of similar plane figures are proportional.*

II. *The areas of similar plane figures are to each other as the squares of their like dimensions.*

III. *The like dimensions of similar plane figures are to each other as the square root of their areas.*

The same principles apply also to the *surfaces* of all similar figures, such as triangles, rectangles, solids, cubes, pyramids, etc.; and to similar *curved surfaces*, as of cylinders, cones, and spheres. Hence,

IV. *The surfaces of all similar figures are to each other as the squares of their like dimensions.*

V. *Their dimensions are as the square roots of their surfaces.*

Examples.

895. 1. A triangular field whose base is 12 ch. contains 2 A. 80 sq. rd. Find the area of a field of similar form whose base is 48 ch.

OPERATION.

$$12^2 : 48^2 :: 2\text{ A. } 80\text{ sq. rd.} : x = 6400\text{ sq. rd.}$$
$$= 40\text{ A., area } (\S\ 894,\ \text{II}).\ \textit{Ans.}$$

2. The side of a square field containing 18 A. is 60 rd. long. Find the side of a similar field that contains $\frac{1}{3}$ as many acres.

OPERATION.

$$18 \text{ A.} : 6 \text{ A.} :: 60^2 : x^2 = 1200 \text{ sq. rd.}$$

$\sqrt{1200} = 34.64$ rd. +, side *Ans.* (§ 894, III).

3. Two circles are to each other as 9 to 16. The diameter of the less being 112 feet, what is the diameter of the greater?

OPERATION.

$$9 : 16 :: 112^2 : x^2 = 3 : 4 :: 112 : x$$

$= 149$ ft. 4 in., diameter *Ans.* (§ 894, II).

4. A peach orchard contains 720 sq. rd., and its length is to its breadth as 5 to 4. What are its dimensions?

OPERATION.

The area of a rectangle 5 by 4 equals 20.

$$\left.\begin{array}{l} 20 : 720 :: 5^2 : x^2 = 900;\ \sqrt{900} = 30 \text{ rd., length} \\ 20 : 720 :: 4^2 : x^2 = 576;\ \sqrt{576} = 24 \text{ rd., width} \end{array}\right\} \textit{Ans.}$$

5. It is required to lay out 283 A. 107 sq. rd. of land in the form of a rectangle, so that the length shall be 3 times the width. Find the dimensions.

6. A pipe 1.5 in. in diameter fills a cistern in 5 hr. Find the diameter of a pipe that will fill the same cistern in 55 min. 6 sec.

7. The area of a triangle is 24276 sq. ft., and its sides are in proportion to the numbers 13, 14, and 15. Find the length of its sides in feet.

8. A field containing 6 A. is laid down on a plan to a scale of 1 in. to 20 ft. How much paper will it cover?

9. If it costs $ 167.70 to inclose a circular pond containing 17 A. 110 sq. rd., find the cost to inclose another $\frac{1}{3}$ as large?

10. If a cistern 6 ft. in diameter holds 80 bbl. of water, what is the diameter of a cistern of the same depth that holds 1200 bbl.?

11. If 63.39 rd. of fence will inclose a circular field containing 2 A., what length will inclose 8 A. in circular form?

Examples on Plane Figures.

896. 1. The area of a triangle is 270 yd., and the perpendicular 45 ft. Find the base.

2. Find the area of a square whose perimeter is the same as that of a rectangle 48 ft. by 28 ft.

3. A rectangle whose length is 3 times its width contains 1323 sq. rd. Find its dimensions.

4. Find the area of an equilateral triangle whose sides are 36 ft.

5. The area of a circle is 7569 sq. ft. Find the length of the side of a square of equal area.

6. How much less will the fencing of 20 A. cost in the square form than in the form of a rectangle whose breadth is $\frac{1}{5}$ the length, the price being $2.40 per rod?

7. A house that is 50 ft. long and 40 ft. wide has a square or pyramidal roof, whose height is 15 ft. Find the length of a rafter reaching from a corner of the building to the vertex of the roof.

8. Find the length of a rafter reaching from the middle of one side.

9. Wishing to know the height of a certain steeple, I measured the shadow of the same on a horizontal plane, $27\frac{1}{2}$ ft.; I then erected a 10-ft. pole on the same plane, and it cast a shadow of $2\frac{2}{3}$ ft. What was the height of the steeple?

10. If the ridge of a building is 8 ft. above the beams, and the building is 32 ft. wide, what must be the length of the rafters?

11. The diameter of a ball weighing 32 lb. is 6 in. What is the diameter of a ball weighing 4 lb.?

12. A general formed his men into a square, that is, an equal number in rank and file, and found that he had 59 men over; and increasing the number in both rank and file by 1 man, he needed 84 more men to complete the square. How many men had he?

13. The shadow of a tree measures 42 ft.; a staff 40 in. in length casts a shadow 18 in. at the same time. What is the height of the tree?

14. What is the diameter of a circular island containing $1\frac{1}{4}$ sq. miles?

15. How many rods more of fencing are required to inclose a square field whose area is 5 acres, than to inclose a circular field having the same area?

16. What is the value of a farm, at $ 75 an acre, its form being a quadrilateral, with two of its opposite sides parallel, one 40 chains and the other 22 chains long, and the perpendicular distance between them 25 chains?

17. A wheel is 3 feet in diameter. How many times will it revolve in going a mile?

18. Find the cost, at 18 cents a square foot, of paving a space in the form of a rhombus, the sides of which are 15 ft., if a perpendicular drawn from one oblique angle will meet the opposite side 9 ft. from the adjacent angle.

19. How much larger is a square which circumscribes a circle 40 rd. in diameter, than a square inscribed in the same circle?

20. What is the value of a piece of land in the form of a triangle, whose sides are 40, 48, and 54 rods, respectively, at the rate of $ 125 an acre?

21. The radius of a circle is 5 feet. Find the diameter of another circle containing 4 times the area of the first.

22. Find the difference in the area of a circle 36 feet in diameter, and the inscribed square.

23. Two steamers after meeting on the ocean part, one sailing north at the rate of 15 miles an hour and the other west at the rate of 20 miles an hour. How far apart will they be in 2 hours?

24. If a pipe 3 inches in diameter discharges 12 hogsheads of water in a certain time, what must be the diameter of a pipe which will discharge 48 hogsheads in the same time?

SOLIDS.

897. A **Solid** or **Body** has three dimensions—length, breadth, and thickness.

The planes which bound it are called its *faces*, and their intersections, its *edges*.

898. The **Convex Surface** of a solid is its surface exclusive of its ends or bases; the **Entire Surface** is the convex surface plus the surface of the bases.

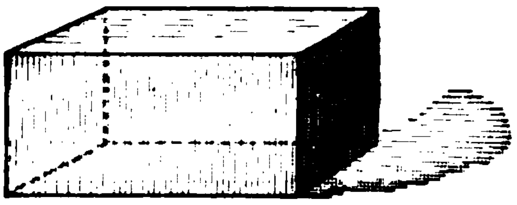
899. The **Volume** of a solid is its *solid contents* in cubic units.

PRISMS AND CYLINDERS.

900. A **Prism** is a solid whose ends are equal and parallel polygons, and its sides parallelograms.

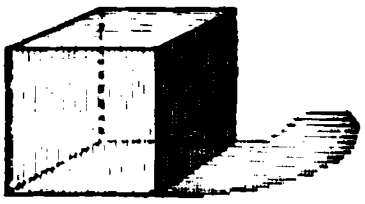
Prisms take their names from the *forms* of their bases, as *triangular*, *quadrangular*, *pentagonal*, etc.

901. The **Altitude** of a prism is the perpendicular distance between its bases.



Parallelopipedon.

902. A **Parallelopipedon** is a prism bounded by six parallelograms, the opposite ones being parallel and equal.



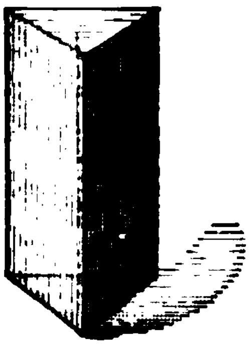
Cube.

903. A **Cube** is a parallelopipedon whose faces are all equal squares.

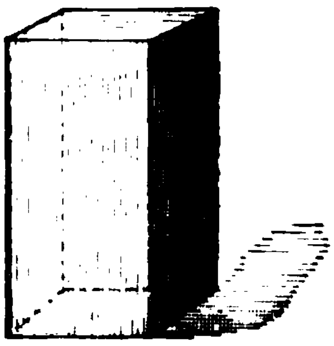
904. A **Cylinder** is a body bounded by a uniformly curved surface, its ends being equal and parallel circles.

1. A cylinder is conceived to be generated by the revolution of a rectangle about one of its sides as an axis.

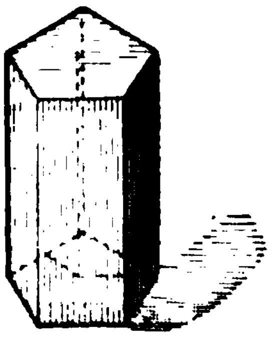
2. The line joining the centers of the bases, or ends, of the cylinder is its *altitude*, or *axis*.



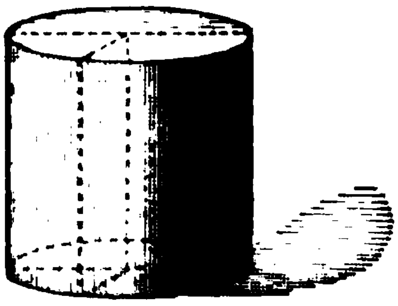
Triangular Prism.



Quadrangular Prism.



Pentagonal Prism.

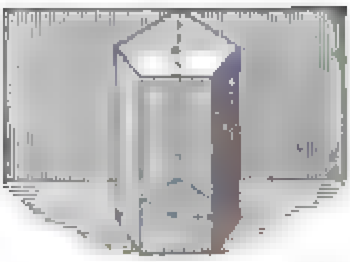


Cylinder.

Examples.

905. To find the convex surface of a prism or cylinder.

1. Find the area of the convex surface of a prism whose altitude is 7 ft. and its base a pentagon, each side of which is 4 ft.



OPERATION.

4 ft. $\times$ 5 = 20 ft., perimeter.

20 $\times$ 7 = 140 sq. ft., convex surface *Ans.*

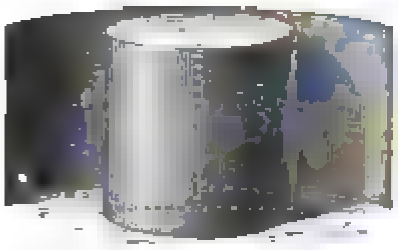
2. Find the area of the convex surface of a triangular prism, whose altitude is $8\frac{1}{2}$ feet, and the sides of its base 4, 5, and 6 feet respectively.

OPERATION.

4 ft. + 5 ft. + 6 ft. = 15 ft., perimeter.

15 $\times$ $8\frac{1}{2}$ = $127\frac{1}{2}$ sq. ft., convex surface *Ans.*

3. Find the area of the convex surface of a cylinder whose altitude is 2 ft. 5 in. and the circumference of its base 4 ft. 9 in.



OPERATION.

2 ft. 5 in. = 29 in; 4 ft. 9 in. = 57 in.

57 $\times$ 29 = 1653 sq. in.

= 11 sq. ft. 69 sq. in. convex surface *Ans.*

RULE. — I. To find the convex surface. — *Multiply the perimeter of the base by the altitude.*

II. To find the entire surface. — *Add the area of the bases or ends.*

4. If a gate 8 ft. high and 6 ft. wide revolves upon a point in its center, what is the entire surface of the cylinder described by it?

5. Find the superficial contents, or entire surface, of a parallelopipedon 8 ft. 9 in. long, 4 ft. 8 in. wide, and 3 ft. 3 in. high.

[redacted]

6. What is the entire surface of a cylinder formed by the revolution about one of its sides of a rectangle that is 6 ft. 6 in. long and 4 ft. wide?

7. Find the entire surface of a prism whose base is an equilateral triangle, the perimeter being 18 ft., and the altitude of the prism 15 ft.

906. To find the volume of any prism or cylinder.

1. Find the volume of a triangular prism, whose altitude is 20 ft., and each side of the base 4 ft.

OPERATION.

The area of the base is 6.928 sq. ft. (§ 861.)
$6.928 \times 20 = 138.56$ cu. ft., volume *Ans.*

2. Find the volume of a cylinder whose altitude is 8 ft. 6 in., and the diameter of its base 3 ft.

OPERATION.

$3^2 \times .7854 = 7.0686$ sq. ft., area of base. (§ 882.)
$7.0686 \times 8.5 = 60.083$ cu. ft., volume *Ans.*

RULE. — *Multiply the area of the base by the altitude.*

3. What is the volume of a parallelopipedon, whose base is 9.8 ft. by 7.5 ft., and its height 5 ft. 3 in.?

4. What is the volume of a log 18 ft. long and $1\frac{1}{2}$ ft. in diameter?

5. What is the height of a block of wood 16^{cm} long and 6.5^{cm} wide which contains $1154.4^{cu\ cm}$?

6. Find the solid contents of a cube whose edges are 6 ft. 6 in.

7. Find the cost of a piece of timber 18 in. square and 40 ft. long at $ .30 a cubic foot.

8. What are the solid contents of a cylinder whose altitude is 15 ft. and its radius 1 ft. 3 in.?

907. To find the three dimensions of a rectangular solid, the volume and the ratio of the dimensions being given.

1. What are the dimensions of a rectangular solid, whose volume is 4480 cu. ft., if its dimensions are to each other as *2, 5, and 7?*

OPERATION.

$\sqrt[3]{4480 \div (2 \times 5 \times 7)} = 4$ ft.; $4 \times 2 = 8$ ft., height; $4 \times 5 = 20$ ft., width; $4 \times 7 = 28$ ft., length.

RULE. — I. *Divide the volume by the product of the terms proportional to the three dimensions, and extract the cube root of the quotient.*

II. *Multiply the root thus obtained by each proportional term; the products will be the corresponding sides.*

2. What are the dimensions of a rectangular box whose volume is 3000 cu. ft., if its dimensions are to each other as 2, 3, and 4?

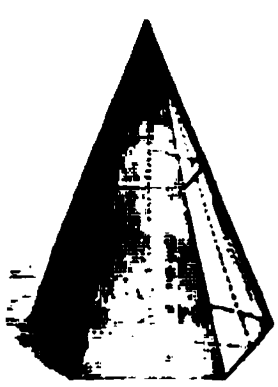
3. A pile of bricks in the form of a parallelopipedon contains 30720 cu. ft., and the length, breadth, and height are to each other as 3, 4, and 5. What are the dimensions of the pile?

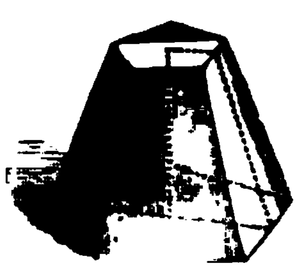
4. Separate 405 into three factors, which shall be to each other as 2, $2\frac{1}{2}$, and 3.

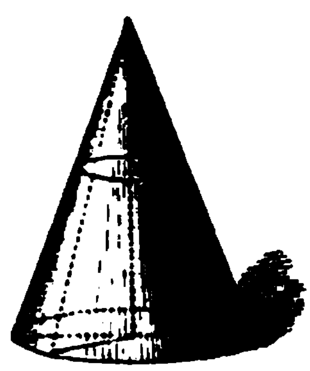
PYRAMIDS AND CONES.

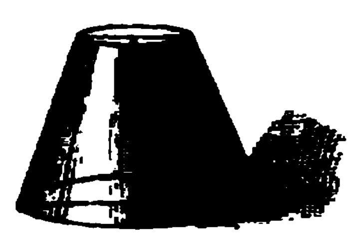
908. A **Pyramid** is a body, having for its base a polygon, and for its other faces three or more triangles, which terminate in a common point called the *vertex*.

Pyramids, like prisms, take their names from their bases, and are called *triangular*, *square*, or *quadrangular*, *pentagonal*, etc.


Pyramid.


Frustum.


Cone.


Frustum.

909. A **Cone** is a body having a circular base, and whose convex surface tapers uniformly to the *vertex*.

A cone is a body conceived to be formed by the revolution of a right-angled triangle about one of its sides containing the right angle.

910. The **Altitude** of a pyramid or of a cone is the perpendicular distance from its vertex to the plane of its base.

911. The **Slant Height** of a *pyramid* is the perpendicular distance from its vertex to one of the sides of the base; of a *cone*, is a straight line from the vertex to the circumference of the base.

912. The **Frustum** of a pyramid or cone is that part which remains after cutting off the top by a plane parallel to the base.

Examples.

913. To find the convex surface of a pyramid or cone.

1. Find the convex surface of a triangular pyramid, the slant height being 16 ft., and each side of the base 5 ft.

OPERATION.

$(5 + 5 + 5) \times \overline{16 \div 2} = 120$ sq. ft., convex surface *Ans.*

2. Find the convex surface of a cone whose diameter is 17 ft. 6 in., and the slant height 30 ft.

OPERATION.

$17.5 \times 3.1416 = 54.978$ ft., circum.; $54.978 \times \overline{30 \div 2}$
$= 824.67$ sq. ft., convex surface *Ans.*

RULE. — I. To find the convex surface. — *Multiply the perimeter or circumference of the base by one half the slant height.*

II. To find the entire surface. — *Add to this product the area of the base.*

3. Find the entire surface of a pyramid whose base is 8 ft. 6 in. square, and its slant height 21 ft.

4. Find the entire surface of a cone the diameter of whose base is 6 ft. 9 in., and the slant height 45 ft

5. Find the cost of painting a church spire, at \$.25 a square yard, whose base is a hexagon 5 ft. on each side, and the slant height 60 ft.

914. To find the volume of a pyramid or of a cone.

1. What is the volume, or solid contents, of a square pyramid whose base is 6 ft. on each side, and its altitude 12 ft.?

OPERATION.

$$\overline{6 \times 6} \times \overline{12 \div 3} = 144 \text{ cu. ft., volume } \textit{Ans.}$$

2. Find the volume of a cone, the diameter of whose base is 5 ft., and its altitude $10\frac{1}{2}$ ft.

OPERATION.

$$(5^2 \times .7854) \times \overline{10\tfrac{1}{2} \div 3} = 68.72\tfrac{1}{4} \text{ cu. ft., volume } \textit{Ans.}$$

RULE. — *Multiply the area of the base by one third the altitude.*

3. Find the solid contents of a cone whose altitude is 24 ft., and the diameter of its base 30 in.

4. What is the cost of a triangular pyramid of marble whose altitude is 9 ft., each side of the base being 3 ft., at \$ $2\frac{1}{4}$ per cu. ft.?

5. Find the volume and the entire surface of a pyramid if its base is a rectangle 80 ft. by 60 ft., and the edges which meet at the vertex are 130 ft.

915. To find the convex surface of a frustum of a pyramid or cone.

1. What is the convex surface of a frustum of a square pyramid whose slant height is 7 ft., each side of the greater base 4 ft., and of the less base 18 in.?

OPERATION.

The *perimeter* of the greater base is 16 ft., of the less 6 ft.

$$\overline{16 + 6} \times \overline{7 \div 2} = 77 \text{ sq. ft., convex surface } \textit{Ans.}$$

RULE. — I. To find the convex surface. — *Multiply the sum of the perimeters, or circumferences, by one half the slant height.*

II. To find the entire surface. — *Add to this product the areas of both ends, or bases.*

2. Find the convex surface of a frustum of a cone whose slant height is 15 ft., the circumference of the lower base 30 ft., and of the upper base 16 ft.

3. How many square yards are there in the convex surface of a frustum of a pyramid, whose bases are heptagons, each side of the lower base being 8 ft., and of the upper base 4 ft., and the slant height 55 ft.?

916. To find the volume of a frustum of a pyramid or cone.

1. Find the volume of a frustum of a square pyramid whose altitude is 10 ft., each side of the lower base 12 ft., and of the upper base 9 ft.

OPERATION.

$$12^2 + 9^2 = 225;\ (225 + \sqrt{144 \times 81}) \times \overline{10 \div 3} = 1110 \text{ cu. ft.}$$

Rule. — *To the sum of the areas of both bases add the square root of the product, and multiply this sum by one third of the altitude.*

2. How many cubic feet are there in the frustum of a cone whose altitude is 6 ft. and the diameters of its bases 4 ft. and 3 ft.?

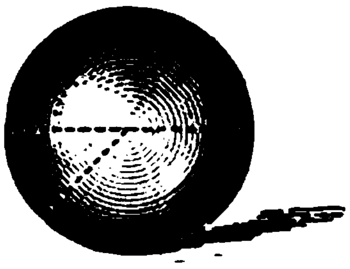
3. How many cubic feet are there in a piece of timber 30 ft. long, the greater end being 15 in. square, and the less 12 in. square?

Spheres.

917. A **Sphere** is a solid bounded by a uniformly curved surface, all the points of which are equally distant from a point within called the *center.*

918. A **Great Circle** is any circle on a sphere whose plane passes through the center of the sphere.

Note. — The circumference of a sphere is any great circle.



919. The **Diameter** of a sphere is a straight line passing through the center of the sphere, and terminated at both ends by its surface.

920. The **Radius** of a sphere is a straight line drawn from the center to any point in the surface.

Examples.

921. To find the surface of a sphere.

1. Find the surface of a sphere whose diameter is 9 in.

OPERATION.

9 in. $\times$ 3.1416 $=$ 28.2744 in., circumference.

28.2744 $\times$ 9 $=$ 254.4696 sq. in., surface Ans.

RULE. — *Multiply the diameter by the circumference of a great circle of the sphere.*

2. What is the surface of a globe 16 in. in diameter?

3. Find the surface of a sphere whose circumference is 31.416 feet.

4. Find the surface of a globe whose radius is 1 foot.

922. To find the volume of a sphere.

1. Find the volume of a sphere whose diameter is 18 in.

OPERATION.

$$18 \text{ in.} \times 3.1416 = 56.5488 \text{ in., circumference.}$$
$$56.5488 \times 18 = 1017.8784 \text{ sq. in., surface.}$$
$$1017.8784 \times \overline{18 \div 6} = 3053.6352 \text{ cu. in., volume } \textit{Ans.}$$

RULE. — *Multiply the surface by $\frac{1}{6}$ of the diameter, or by $\frac{1}{3}$ of the radius.*

2. Find the volume of a globe whose diameter is 30 in.

3. Find the solid contents of a globe whose radius is 5 yd.

4. Find the volume of a globe whose circum. is 31.416 ft.

923. SUMMARY OF SPHERES.

1. The *Surface* $= \begin{cases} \text{Circumference} \times \text{its diam.} \\ \text{Radius}^2 \times 12.5664. \\ \text{Diameter}^2 \times 3.1416. \\ \text{Circumference}^2 \times .3183. \end{cases}$

2. The *Volume* $= \begin{cases} \text{Surface} \times \frac{1}{6} \text{ its diameter.} \\ \text{Radius}^3 \times 4.1888. \\ \text{Diameter}^3 \times .5236. \\ \text{Circumference}^3 \times .0169. \end{cases}$

3. The *Diameter* $= \begin{cases} \sqrt{\text{Of surface}} \times .5642. \\ \sqrt[3]{\text{Of volume}} \times 1.2407. \end{cases}$

4. The *Circumference* $= \begin{cases} \sqrt{\text{Of surface}} \times 1.77255. \\ \sqrt[3]{\text{Of volume}} \times 3.8978. \end{cases}$

5. The *Radius* $= \begin{cases} \sqrt{\text{Of surface}} \times .2821. \\ \sqrt[3]{\text{Of volume}} \times .6204. \end{cases}$

6. The *Side of an inscribed cube* $= \begin{cases} \text{Radius} \times 1.1547. \\ \text{Diameter} \times .5774. \end{cases}$

Similar Solids.

924. **Similar Solids** are such as have the same *form*, and differ from each other only in volume.

Principles. — I. *The volumes of similar solids are to each other as the cubes of their like dimensions.*

II. *The like dimensions of similar solids are to each other as the cube roots of their volumes.*

Examples.

1. If the volume of a ball 3 in. in diameter is 27 cu. in., what is the volume of a ball 7 in. in diameter?

OPERATION.

$3^3 : 7^3 :: 27$ cu. in. $: x = 343$ cu. in., volume *Ans.*

2. If the diameter of a ball whose volume is 27 cu. in. is 3 in., what is the diameter of a ball whose volume is 343 cu. in.?

OPERATION.

$27 : 343 :: 3^3 : x^3 = 343$; $\sqrt[3]{343} = 7$ in., diameter *Ans.*

Examples on Solids.

925. 1. What is the edge of a cube whose entire surface is 1050 sq. ft., and what is its volume?

2. What must be the inner edge of a cubical bin to hold 1250 bushels of wheat?

3. How many globes 4 in. in diameter are equal to one whose diameter is 12 in.?

4. How many gallons will a cistern hold, whose depth is 7 ft., the bottom being a circle 7 ft. in diameter and the top 5 ft. in diameter?

5. What is the value of a stick of timber 24 ft. long, the larger end being 15 in. square, and the less 6 in., at 28 cents a cubic foot?

6. If a cubic foot of iron were formed into a bar $\frac{1}{2}$ an inch square, without waste, what would be its length?

7. How many barrels of $31\frac{1}{2}$ gal. will a cistern hold that is 8.3 ft. in diameter, and 7 ft. deep?

8. If a log 18 ft. long and 3 ft. in diameter is hewn square, how many cubic feet does it contain?

9. Find the volume of a cube, the area of whose entire surface is 7 sq. ft. 6 sq. in.

10. If a marble column 10 in. in diameter contains 27 cu. ft., what is the diameter of a column of equal length that contains 81 cu. ft.?

11. Supposing the earth to be a perfect sphere 7912 miles in diameter, what is its volume in cubic miles?

12. How many board feet are there in a post 11 ft. long, 9 in. square at the bottom, and 4 in. square at the top?

13. The surface of a sphere is the same as that of a cube, the edge of which is 12 in. Find the volume of each.

14. The contents of a cubical block of marble are 4913 cu. ft. Find the superficial contents or surface.

15. Find the dimensions of a bin that holds 450 bu. of grain, if the width and depth are equal, and the length 3 times the width.

16. A ball 4.5 in. in diameter weighs 18 oz. avoir. What is the weight of a ball of the same density that is 9 in. in diameter?

17. In what time will a pipe supplying 6 gal. of water a minute fill a tank in the form of a hemisphere that is 10 ft. in diameter?

18. If the altitude of a cone that weighs 640 lb. is 8 ft., what is the altitude of a similar cone that weighs 270 lb.?

19. If a stack of hay 8 ft. high weighs 8 cwt., what is the weight of a similar stack that is 24 ft. high?

20. The diameter of a cistern is 8 ft. What must be its depth to contain 75 hhd. of water?

21. If a cable 3 in. in circumference supports a weight of 2500 lb., what must be the circumference of a cable that will support 4960 lb.?

22. How many bushels are there in a heap of grain in the form of a cone, whose base is 8 ft. in diameter and altitude 4 ft.?

MISCELLANEOUS EXAMPLES.

926. 1. How many thousand shingles will cover both sides of a roof 36 ft. long, whose rafters are 18 ft. in length?

2. From $\frac{2}{7}$ of $\frac{4}{7}$ of $\frac{1}{8}$ of 70 miles, subtract .73 of 1 mi. 3 fur.

3. What number is that from which if $7\frac{1}{2}$ is subtracted $\frac{2}{3}$ of the remainder is $91\frac{1}{2}$?

4. What part of 4 is $\frac{4}{9}$ of 6?

5. What number increased by $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$ of itself equals 125?

6. What is the hour, when the time past noon is equal to $\frac{2}{3}$ of the time to midnight?

7. If \$240 gains \$5.84 in 4 mo. 26 da., what is the rate per cent?

8. If 24 men in 189 da., working 10 h. a day, dig a trench $33\frac{3}{4}$ yd. long, $2\frac{2}{3}$ yd. deep, and $5\frac{1}{4}$ yd. wide, how many hours a day must 217 men work, to dig a trench $23\frac{1}{4}$ yd. long, $2\frac{1}{3}$ yd. deep, and $3\frac{2}{3}$ yd. wide, in $5\frac{1}{2}$ days?

9. What is the difference between the interest and the bank discount of \$450 at 5%, for 6 yr. 10 mo.?

10. A younger brother received \$6300, which was $\frac{7}{8}$ as much as his elder brother received. How much did both receive?

11. Reduce $.\dot{7}$, $.\dot{8}\dot{8}$, $.\dot{7}2\dot{7}$, $.9\dot{1}32\dot{5}$ to their equivalent common fractions.

12. A man by selling a lot of goods for \$438, loses 10%. *How much* should the goods have been sold for to gain $12\frac{1}{2}$%?

$330. What was the capital of each

15. Henry Truman purchased co months' credit, as follows: Aug. 27, 31, 150 bu. @ $.40; Sept. 7, 500 bu. bu. @ $.42; Sept. 25, 250 bu. @ $.40. due per average?

16. A, B, and C can do a job of wor in 24 da., and A in 34 da. In what tin

17. If a man travels 7 mi. the first d increasing his journey 4 mi. each day he travel, and how far?

18. What is the difference between count of $2500, payable in 90 days at 7

19. I sold $\frac{1}{2}$ of a lot of lumber for w per cent did I gain on the part sold?

20. If $500 gains $50 in 1 yr., in wl $60?

21. A dealer received an invoice which was broken. At what per cent remainder be sold to clear 25% on the i

22. The sum of two numbers is 365,

26. A boy 14 years old is left an annuity of \$250, which is deposited in a savings bank at 6 %, interest payable semi-annually. How much will he be worth when of age?

27. If a boy buys peaches at the rate of 5 for 2 cents, and sells them at the rate of 4 for 3 cents, how many must he buy and sell to make a profit of \$4.20?

28. A fox 30 rd. ahead of a hound runs 19 ft. while the hound runs 20 ft. How far will the fox run before he is overtaken?

29. How long is a rope from the top of a pole 50 ft. high to the top of another 20 ft. high, the poles being 16 ft. apart?

30. Three men paid \$100 for a pasture; A put in 9 horses, B 12 cows for twice the time, and C some sheep for $2\frac{1}{2}$ times as long as B's cows; C paid one half the cost. How many sheep had he, and how much did A and B each pay, if 6 cows eat as much as 4 horses, and 10 sheep as much as 3 cows?

31. A fountain has 4 receiving pipes, A, B, C, and D; A, B, and C will fill it in 6 hours, B, C, and D in 8 hours, C, D, and A in 10 hours, and D, A, and B in 12 hours; it has also 4 discharging pipes, W, X, Y, and Z; W, X, and Y will empty it in 6 hours, X, Y, and Z in 5 hours, Y, Z, and W in 4 hours, and Z, W, and X in 3 hours. Suppose the pipes are all open and the fountain full, in what time will it be emptied?

32. Find the cost, at 18 cents a square foot, of paving a space in the form of a rhombus, if its sides are 15 ft., and a perpendicular drawn from one oblique angle will meet the opposite side 9 ft. from the adjacent angle.

33. Four boys are playing at hare and hounds, the two representing hares being 45 ft. in advance of the two representing hounds. Each hound's leap covers $6\frac{1}{2}$ ft. of ground, and each hare's leap $4\frac{1}{2}$ ft., but the hares take 4 leaps to the hounds' 3. In how many leaps will the hounds overtake the hares?

34. Five men can do a piece of work in 9 da. How soon after beginning must they be joined by two more so as to complete the work in 8 da.?

35. Four men contracted to do a certain piece of work for \$8600; the first employed 28 laborers 20 da., 10 h. a day; the second, 25 laborers 15 da., 12 h. a day; the third, 18 laborers 25 da., 11 h. a day; and the fourth, 15 laborers 24 da., 8 h. a day. How much should each contractor receive?

36. If I exchange 75 railroad bonds of \$500 each, at 64, for bank stock at 105, how many shares of \$100 each shall I receive?

37. A flour merchant bought 120 bbl. of flour for \$660, paying \$5.75 for first quality and \$5 for second quality. How many barrels were first quality?

38. Two mechanics work together; for 15 days' work of the first and 8 days' work of the second they receive \$61, and for 6 days' work of the first and 10 days' work of the second they receive \$38. How much does each man earn?

39. A dairyman took some butter to market, for which he received \$49, receiving as many cents a pound as there were pounds. How many pounds were there?

40. A mechanic received \$2 a day for his labor, and paid \$4 a week for his board; at the expiration of 10 weeks he had saved \$72. How many days did he work, and how many was he idle?

41. To what would \$250, deposited in a savings bank, amount in 10 yr., interest being allowed semiannually at 6% per annum?

42. How much water is there in a mixture of 100 gal. of wine and water, worth \$1 per gal., if 100 gal. of the wine cost \$120?

43. If a pipe 3 in. in diameter will discharge a certain quantity of water in 2 h., in what time will 3 two-inch pipes discharge 3 times the quantity?

44. William Jones & Co. become insolvent and owe \$8100. Their assets amount to \$4981.50. What per cent of their indebtedness can they pay, allowing the assignees $2\frac{1}{2}$% on the amount distributed for their services?

45. I shipped a car load of cattle to Boston, and offered them for sale at 25% advance on the cost; but the market being dull I sold for 14% less than my asking price, and gained thereby $170. How much did the cattle cost; for how much did they sell; and what was my asking price?

46. What must be the dimensions of a cubical cistern to hold 2000 gallons?

47. A vessel having sailed due south and due east on alternate days, was found, after a certain time, to be 118.794 miles southeast of the place of starting. What distance had she sailed?

48. If $34\frac{1}{2}$ bu. of corn are equal in value to 17 bu. wheat, 9 bu. of wheat to $59\frac{1}{2}$ bu. of oats, and 6 bu. of oats to 42 lb. of flour, how many bushels of corn will purchase 5 bbl. of flour?

49. If stock bought at 92 will pay 7% on the investment, at what rate should it be bought to pay 10%?

50. A merchant in New York paid $2000 for a bill of exchange of £400 to remit to Liverpool. What was the rate of exchange?

51. A, B, and C start from the same point, to travel around a lake 84 miles in circumference. A travels 7 miles, and B 21 miles a day in the same direction, and C 14 miles in an opposite direction. In how many days will they all meet?

52. The exact solar year is greater than 365 days by about $\frac{20929}{86400}$ of a day. Find approximately how often leap year should come, or one day be added to the common year, in order to keep the calendar right.

53. What per cent shall I gain by purchasing goods on 6 mo., and selling them immediately for cash at cost, money being worth 7%?

54. What sum must a man save annually, commencing at 21 years of age, to be worth $30000 when he is 50 years old, his *savings being* invested at 5% compound interest?

55. If 6 apples and 7 peaches cost 33 cts., and 10 apples and 8 peaches cost 44 cts., what is the price of one of each?

56. Three persons are to share $10000 in the ratio of 3, 4, and 5, but the first dying, it is required to divide the whole sum equitably between the other two. What are the shares of the other two?

57. If 50 bbl. of flour in Chicago are worth 125 yd. of cloth in New York, and 80 yd. of cloth in New York are worth 6 bales of cotton in Charleston, and 13 bales of cotton in Charleston are worth $3\frac{1}{2}$ hhd. of sugar in New Orleans, how many hogsheads of sugar in New Orleans are worth 1500 bbl. of flour in Chicago?

58. Seven men all start together to travel the same way round an island 120 mi. in circumference, and continue to travel until they all come together again. They travel 5, $6\frac{1}{4}$, $7\frac{1}{3}$, $8\frac{1}{4}$, $9\frac{1}{2}$, $10\frac{1}{4}$, and $11\frac{1}{4}$ mi. a day respectively. In how many days will they all be together again?

59. If stock bought at 105 will pay 6% on the investment, what per cent will it pay if bought at 85?

60. A man in dividing his estate among his sons gave A $9 as often as B $5, and C $3 as often as B $7. C's share was $3862.50. What was the value of the whole estate?

61. A drover sold some oxen at $28, cows at $17, and sheep at $7.50 per head, and received $749 for the lot. There were twice as many cows as oxen, and three times as many sheep as cows. How many were there of each kind?

62. For what sum must a vessel, valued at $25000, be insured, so that in case of its loss, the owners may recover both the value of the vessel and the premium of 24%?

63. What will be the difference in the expense of fencing two fields of 25 acres each, one square, and the other in the form of a rectangle, whose length is twice its breadth, the fence costing $.62$\frac{1}{2}$ a rod?

64. At what time between 5 and 6 o'clock are the hour and minute hands of a watch exactly together?

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65. A boy agreed to work for a merchant for 20 weeks, on condition that he should receive $ 20 and a coat. At the end of 12 weeks the boy stopped working, and it was found that he was entitled to $ 9 and the coat. What was the value of the coat?

66. An irregular piece of land, containing 540 A. 36 sq. rd., is exchanged for a square piece containing the same area. What is the length of one of its sides? If divided into 42 equal squares, what will be the length of the side of each?

67. A general, forming his army into a square, had 284 men remaining; but increasing each side by one man, he wanted 25 men to complete the square. How many men had he?

68. Divide $ 3648 among 3 persons, so that the share of the first to that of the second shall be as 7 to 9, and of the first to the third as 3 to 4.

69. If a lot of land, in the form of an oblong or rectangle, contains 6 A. 132 sq. rd., and its length is to its width as 21 to 13, what are its dimensions; and how many rods of fence will be required to inclose it?

70. Five persons are employed to build a house. A, B, C, and D can build it in 13 days; A, B, C, and E in 15 days; A, B, D, and E in 12 days; A, C, D, and E in 19 days; and B, C, D, and E in 14 days. In how many days can all together build it; and which one could do the work alone in the shortest time?

71. Divide $ 500 among 3 men, in such a manner that the share of the second may be $\frac{1}{2}$ greater than that of the first, and the share of the third $\frac{1}{2}$ greater than that of the second.

72. A and B engage in trade; A puts in $ 5000, and at the end of 4 mo. takes out a certain sum. B puts in $ 2500, and at the end of 5 mo. puts in $ 3000 more. At the end of the year A's gain is $ 1066$\frac{2}{3}$, and B's is $ 1333$\frac{1}{3}$. What sum did A take out at the end of 4 mo.?

73. What sum of money, with its semiannual dividends of 5% invested with it, will amount to $ 12750 in 2 yr.?

74. If a piece of silk cost \$.80 per yard, at what price shall it be marked, that the merchant may sell it at 10% less than the marked price, and still make 20% profit?

75. A merchant bought 20 pieces of cloth, each piece containing 25 yd. at \$ $4\frac{3}{8}$ per yard on a credit of 9 mo.; he sold the goods at \$ $4\frac{5}{8}$ per yard on a credit of 4 mo. What was his net cash gain, money being worth 6%?

76. A owes B \$1200, to be paid in equal annual payments of \$200 each; but not being able to meet these payments at their maturities, and having an estate 10 years in reversion, he arranges with B to wait until he enters upon his estate, when he is to pay B the whole amount, with 8% compound interest. What sum will B then receive?

77. A man who was entitled to a perpetuity of \$3000 a year, provided in his will that, after his decease, his oldest son should receive it for 10 yr., then his second son for the next 10 yr., and a literary institution forever afterward. What was the value of each bequest at the time of his decease, allowing compound interest at 6%?

78. B has 3 teams engaged in transportation; his horse team can perform the trip in 5 days, the mule team in 7 days, and the ox team in 11 days. Provided they start together, and each team rests a day after each trip, how many days will elapse before they all rest the same day?

79. A man bought a farm for \$4500, and agreed to pay principal and interest in 4 equal annual installments. How much was the annual payment, interest being 6%?

80. A man desires to set out a rectangular orchard of 864 trees, so placed that the number of rows shall be to the number of trees in a row, as 3 to 2. If the trees are 7 yards apart, how much ground will the orchard occupy?

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